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Complex analytic dynamics: A modern perspective of iteration in $\mathbb{C}$

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Abstract

In this paper an exhaustive survey of complex analytic dynamics is presented, highlighting in the process significant evolutionary trends the subject has undergone right from the local, down to the global theories. The exponential function $E_\lambda(z) = \lambda e^z$ is then used as a classical example of an entire transcendental function to illustrate new phenomenon of modern perspectives of iteration in $\mathbb{C}$.

Keywords
embeddable, entire transcendental function, Fatou set, Julia set, set of non normality, periodic point, wandering domain, fundamental domain,

1.0 Introductory Survey

Let $f(z)$ be a function of a complex variable z and α a fixed point of $f(z)$. The main concern of the local theory is to determine the rate of convergence of the sequence of iterates $\{f^n(z)\}$. $n = 0,1,2,\ldots$ in some neighborhood $N_{(\alpha)}$ of α. The important result in this part of the theory can be found in the papers of Baker [1, 2], Fatou [10, 11, 12], Julia [13], Siegel [19], and Whittington [22]. The local theory of natural iteration was later generalized to that of continuous iteration and we present here some of the known relevant results. Let $f(z)$ be given by the power series

$$f(z) = a_1 z + a_2 z^2 + \cdots, |z| < R \tag{1.1}$$

So that $z = 0$ is a fixed point of multiplier a_1, then the n^{th} iterate $f^n(z)$ of $f(z)$ is given as

$$f^n(z) = b_1 z + b_2 z^2 + \cdots \tag{1.2}$$

Convergent in some neighborhood $|z| < R_1$ of $z = 0$. Here $b_1 = {a_1}^n$ and b_n is a function of the a_i's. We now replace n by an arbitrary real or complex number σ and define formally the continuous iterate $f^\sigma(z)$ of $f(z)$ by

$$f^n(z) = a_1(\sigma) z + a_2(\sigma_2) z^2 + \cdots + a_n(\sigma) z^n + \cdots \tag{1.3}$$

Where the coefficients $a_n(\sigma)$ are uniquely determined from the conditions

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$$a_1(\sigma) = a_1^0 \text{ and } f_\sigma \circ f = f \circ f_\sigma \quad (1.4)$$

Unlike the natural iterates f^n, the continuous iterates f^σ,may not converge at all. Hence it is natural to investigate the question of convergence of the continuous iterates. This question of convergence of continuous iterates near a fixed points of multiplier a_1 such that $0 < |a_1| < 1 \; or \; |a_1| > 1$ was completely answered by Koenigs (1984) by the use of the Schroder functional equation

$$\phi(f^n(z)) = a_1{}^n \phi(z) \quad (1.5)$$

where ϕ is given by the (Schroder series)

$$\varphi(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (1.6)$$

convergent in some neighborhood of $z = 0$, and satisfying

$$\phi(f(z)) = a_1 \varphi(z) \quad (1.7)$$

so that

$$f^n(z) = \varphi_{-1}\{a_1{}^n \phi(z)\} \quad (1.8)$$

in some neighborhood of $z = 0$.

One can replace n in (1.8) by an arbitrary real or complex number σ and for suitable determinations of $a_1{}^\sigma$, the expression

$$f^\sigma(z) = \varphi_{-1}\{a_1{}^\sigma \phi(z)\} \quad (1.9)$$

generalizes $f^n(z)$ to a family of continuous iterates of $f^\sigma(z)$ which are permutable in the sense that

$$f^\sigma(f^\tau(z)) = f^\tau(f^\sigma(z)) = f^{\sigma+\tau}(z) \quad (1.10)$$

The question of convergence near an indifferent fixed point proved very difficult. Cremer [6] and Siegel [19] have shown that the convergence of the series obtained when $a_1 = \exp(i\theta)$ depends on a number of theoretical properties of θ. If a_1 is not a root of unity, then the Schroder function $\varphi(z)$ of (1.9) can be calculated formally, but it need not converge.

Cremer showed that if

$$\lim_{n\to\infty} \text{Inf} \, |a_1{}^n - 1| = 0 \quad (1.11)$$

there exist functions with fixed points of multiplier a_1 for which the Schroder series (1.6) diverges. On the other hand Siegel showed that if

$$\log|a_1{}^n - 1| = O(\log n) \quad (n \to \infty) \quad (1.12)$$

then the Schroder series (1.6) converges for every $f(z)$ with $f(0) = 0$ and $f'(0) = a_1$. However it is generally not possible to formulate the Schroder function if a_1 is not a root of unity but $|a_1| = 1$. In this case the Schroder function exists if and only if the n^{th} iterate of $f(z)$ is the identity function.

The case of natural iterates $f^n(z)$ when $a_1 = 1$ has been studied by Fatou [10, 11, 12]. In this case also the continuous iterates analytic at $z = 0$ are not provided for by the Schroder functional equation.

For $a_1 = 1$ are proceed as follows: we represent $f^\sigma$by the formal power series

$$f^\sigma(z) = z + a_2 \sigma z^2 + \sum_{n=3}^{\infty} b_n(\sigma) z^n \quad (1.13)$$

where $b_n(\sigma)$ are polynomials in σ. The existing literature deals with the question of determining the values of σ for which the formal power series (1.13) converges for a given z.

The following theorem of Baker is of fundamental importance.
***Theorem* 1.1** Baker [1].

If R is the set of values corresponding to the convergent members of the family (1.13) then R has one of the following forms.

(*i*) The point $\sigma = 0$

(*ii*) $\{n\ \sigma_0\}, n = 0,\ \pm 1,\ \pm 2,\ \cdots,$ and $\sigma_0 \neq 0$

(*iii*) $\{m\ \sigma_0 + n\ \sigma_1\}$, where $m = 0,\ \pm 1,\ \pm 2, \cdots,\ n = 0,\ \pm 1,\ \pm 2,\ \cdots,\ \sigma_0 \neq 0,$ $\sigma_1 \neq 0$ *and* ${}^{\sigma_0}/{\sigma_1}$ *is not real*

(*iv*) The whole plane

Liverpool [15] improved the above theorem by proving the case (*iii*) cannot occur at all.

We shall afford the following definition which is fundamental to the theorem that follows:-

***Definition* 1.1**

A function *f*(*z*) of the complex variable z is called embeddable if the case (*iv*) of theorem 1.1 above occurs otherwise f is called non-embeddable.

The function $f(z) = {}^{z}/{(1+z)}$ provides an easy example of an embeddable function. Since In this case $f^{\sigma}(z) = {}^{z}/{(1+\sigma z)}$ converges for all values of σ. The following theorem gives examples of classes of non- embeddable functions.

***Theorem* 1.2** Baker [3]

If $f(z) = e^{z} - 1$ then $f^{\sigma}(z)$ converges only for integral values of σ.

***Theorem* 1.3** Baker [4]

If $f(z)$ is a meromorphic function with an expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n, \qquad a_2 \neq 0$$

near the origin, then $f(z)$ is non-embeddable except when

$$f(z) = {}^{z}/{(1+az)}$$

Yet other classes of non-embeddable functions can be found in Liverpool [15]. The global theory of iteration developed by Fatou and Julia on the other hand deals with the family $\{f^n(z)\}$ of natural iterates of the function of complex variable. This theory is also known as the Fatou - Julia theory – in which they considered only rational and entire functions. Radstrom examined how far the theory of Fatou and Julia can be extended to other classes of functions. He proved that the Fatou-Julia theory exists for the rational and entire functions considered by Fatou and Julia and precisely for one other class of functions; functions having exactly one pole and one essential singularity in the extended plane.

2.0 Advances in complex analytic dynamics

We shall now review the successes achieved over the years in an attempt to resolve some of the problems and open questions encountered as the subject of complex dynamics advanced into the century. Indeed it is already an established fact that complex analytic dynamics has under gone rapid development in recent years.

After a period of relative dormancy, the field was rejuvenated in 1980 because of some intriguing computer graphics, images of the Mandelbrot set as well as major new mathematical advances due to Douady and Hubbord [9] and Sullivan [20, 21] among others.

The field of complex analytic dynamics has experienced two relatively short periods of vigorous growth. The field traces its origin to the global theory which began in the late nineteenth and early twentieth century during which time; mathematicians such as Leau, Schroder, Keonigs, and Bottcher became interested in the local behavior of complex functions under iteration.

During the period of 1918-20, a dramatic shift in emphasis occurred due primarily to the pioneering efforts of two French mathematicians Gaston Julia and Pierre Fatou. Instead of considering only local dynamic behavior, Julia and Fatou took a more global point of view. Away from the fixed points they found quite different dynamical behavior. Sometimes the results of iteration are quite tame or stable: at other times these iterations behaved in dynamically different fashion - what we now call chaotic behavior. It is in their honor that we now call the stable region for a complex dynamical system the Fatou set, while the chaotic region is known as the Julia set. In a series of remarkable papers between 1918-20, Fatou and Julia succeeded in describing many of the properties of both of these sets for rational maps. However in attempting to classify all possible dynamics on the Fatou set, they reached an impasse. They could not rule out the possibility that the dynamics included wandering domains, which they expected would not occur, nor could they prove the existence of what are known as Siegel disks which they expected would occur. With these two road blocks in the way, work in complex analytic dynamics was slowed down and there was not much activity in the field for about fifty years.

There were two notable events which occurred during this period of dormancy. Siegel [19] showed that indeed Siegel disks could in fact exist in complex dynamical systems. This brought the classification of stables regions one step closer to completion. Later, Baker extended much of the work of Fatou and Julia to other classes of functions, showing along the way that other types of stable behavior could occur for entire and meromorphic functions.

The second major period of activity began when Mandelbrot [18], first used computer graphics to explore complex dynamics. His discovery of the Mandelbrot set prompted many mathematicians to re investigate this field. In quick succession, Sullivan introduced the use of quasi-conformal maps into the subject. This allowed him to prove his “No wandering domains theorem”, which essentially completed the classification of stable dynamics of rational maps began by Fatou and Julia. At the same time Douady and Hubbord opened new vistas in the field by considering the parameter space for quadratic polynomials. They developed a technique which enabled them to classify less or more completely all possible types of quadratic dynamics. The work of Blanchard [5] is indeed a key source material in this direction as well as the results of Liverpool and Yuguda [16 and 17] both of which link up iteration theory and complex analytic dynamics.All of these and the ground work of Devaney [8], formed the initial motivation for this work.

2.1 Complex dynamics and entire transcendental functions

Our interest in this section is to illustrate some of the new phenomenon that occurs in the iteration of rational maps and entire transcendental functions. The main differences arise because of the point at ∞. For polynomials, this point is a super attracting fixed point. For rational maps, ∞ is dynamically the same as any other point in the Riemann sphere. But for entire transcendental functions, ∞ is an essential singularity. By the Picard theorem, any plane neighborhood of ∞ is mapped by an entire function over the entire plane, missing at most one point, and taking any other value infinitely often. Thus the point at ∞ injects a tremendous amount of "hyperbolicity" into the dynamics and results in substantial differences in both the dynamics on and the geometry of the Julia sets for these maps.

In this paper we will concentrate on the Julia sets of the entire transcendental function, λe^z, where $\lambda > 0$ is a real parameter. It is to be noted that entire functions share many of the dynamical properties of polynomial or rational maps, but there are some major differences. For example, the Julia sets of these maps have the interesting property that they may explode as the parameter λ is varied. That is, the Julia set undergoes sudden dramatic changes at certain critical parameter values. In the exponential case, one such explosion occurs when $\lambda = \frac{1}{e}$. For instance for λ-values less than this critical value, the Julia set occupies a relatively small, nowhere dense subset of the right half plane, but when $\lambda > \frac{1}{e}$ the Julia set is the whole plane.

This means that the set of chaotic orbits changes discontinuously at this parameter value. This type of bifurcation cannot happen in the families of polynomial or rational maps. Our major goal in this paper is to describe this explosion in detail.

In the preceding section we present some preliminary notions from complex analysis and dynamical systems theorem. We also give a precise definition of Julia sets and formulate some of its basic properties. We then describe the Julia set of λe^z before the explosion occurs and then prove that the Julia set is a nowhere dense subset of the right half plane when $\lambda < \frac{1}{e}$ following which we can concentrate on the case $\lambda > \frac{1}{e}$ where the Julia set is the whole plane. We also present several alternative characterizations of the Julia set in the two cases highlighted.

2.1.1 Preliminaries

Our ideal point of departure before looking at the specific case of λe^z would be first to consider some basic definitions from dynamical systems theory. First, a fixed point for a function *f*, as the name implies is a point z such that $f(z) = z$. The point z is periodic if there exist an integer *n* such that $f^n(z) = z$. We define the smallest positive n as the period of z and note immediately that any fixed point, z is periodic with period one, and in fact $f^n(z) = z$ for all n.

Of particular interest to us as we consider Julia sets will be the behavior of orbits nearby certain periodic points. We first distinguish several different types of periodic points.

First, if z_0 is a periodic point of period n, and $0 < |(f^n)'(z_0)| < 1$, then z_0 is said to be an attracting periodic point. In the special case when $|(f^n)'(z_0)| = 0$, we call z_0 a super attracting periodic point.

If z_0 is such that $|(f^n)'(z_0)| > 1$ then z_0 is said to be a repelling periodic point. The intermediate case, $|(f^n)'(z_0)| = 1$, gives indifferent periodic points. As we would expect, points nearby an attracting fixed point tend, upon iteration, towards that point. In the case of repeller, points nearby the fixed point eventually escape from that point upon iteration. These points can be made more precise into a theorem which we proceed to give.

***Theorem* 2.1.1**

If z_0 is an attracting fixed point, there is an open disk $\cup$ about z_0 such that if $z \in \cup$, then $\lim_{n\to\infty}(f^n)(z) = z_0$.

Proof

Since z_0 is an attracting fixed point, there exists $\delta > 0$ and $\mu < 1$ such that if $|z - z_0| < \delta$, then $|f'(z)| < \mu$. Hence it follows that if $z_1 z_2 \in B_\delta(z_0)$, the ball of radius δ about z_0, then

$$|f(z_1) - f(z_2)| < \mu|z_1 - z_2| < |z_1 - z_2|$$

Therefore, $f\big(B_\delta(z_0)\big) \subset B_{\mu\delta}(z_0)$. By the contraction mapping principle, f is a contraction in $B_\delta(z_0)$, and so all points in the ball tend to the fixed point z_0 under iteration.

In the above proof, the set $D = B_\delta(z_0) - f\big(B_\delta(z_0)\big)$ is called the fundamental domain. This means that if $z \in B_\delta(z_0), z \neq z_0$, then there is a unique $w \in D$ and $n \geq 0$ such that $f^n(w) = z$. This means that each forward orbit in $B_\delta(z_0)$ which is not the fixed point passes through D exactly once.

To see why this is true, we note that, as above,

$$f^n\big(B_\delta(z_0)\big) \subset B_{\mu^n\delta}(z_0).$$

Hence there is a smallest integer $j \geq 0$ such that $z \in f^j\big(B_\delta(z_0)\big)$ but $z \notin f^{j+1}\big(B_\delta(z_0)\big)$. Then we may let $w \in D$ be such that $f^j(w) = z$ and $n = j$ (see Figure 2.1)

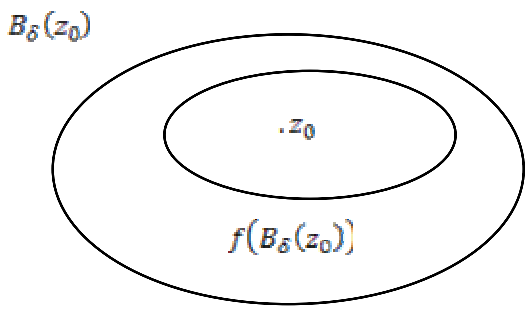

Figure 2.1: $B_\delta(z_0) - f\big(B_\delta(z_0)\big)$ is a fundamental domain

We call the set of all points whose orbits tend to a given fixed point the basin of attraction of that point. Clearly, the basin of attraction is an open set.

In the case of a repelling fixed point, nearby points eventually escape from the repeller upon iteration. In contrast to the previous theorem, we state

***Theorem* 2.1.2**

If z_0 is a repelling fixed point then there exists a disk $\cup$ about z_0 such that if $z \in \cup\ (z \neq z_0)$, then there exists $k > 0$ such that $f^k(z) \notin \cup$

Proof

Since z_0 is a repelling fixed point, we have $|f'(z_0)| > 1$. Hence by the inverse mapping theorem, there exists a neighborhood of z_0 on which a branch of f^{-1} exists. We compute

$$\big|(f^{-1})'(z_0)\big| = 1/\big|f'\big(f^{-1}(z_0)\big)\big| = 1/|f'(z_0)| < 1.$$

And we conclude that z_0 is an attracting fixed point for f^{-1} and hence by our previous theorem there exists $\delta > 0$ such that if $z \in B_\delta(z_0)$ then $f^{-1}(z) \in B_\delta(z_0)$ and $(f^{-1})^n(z) \to z_0$.

Now assume that for any neighborhood $\cup$ of z_0, there exists $z^* \neq z_0$ which satisfies $f^k(z^*) \in \cup$ for all $k \geq 0$. We may assume that $\cup = f^{-1}(B_\delta(z_0))$ for δ sufficiently small. As above, $B_\delta(z_0) - f^{-1}(B_\delta(z_0))$ is a fundamental domain for f^{-1}. Hence there exists $w \in B_\delta(z_0) - f^{-1}(B_\delta(z_0))$ and $n > 0$ such that $(f^1)^n(^w) = z^n$. But then $f^n(z^*) = w \notin \cup$. This contradiction establishes the result.

In order to give a more precise definition of the Julia set, we now shift our attention to some important concepts from classical complex analysis. Let $\mathcal{F}$ be a family of analytic functions defined on an open $U \subset \mathbb{C}$. For our purpose, $\mathcal{F}$ will usually be the family of iterates of a given complex function. The family $\mathcal{F}$ is said to be normal on U if every sequence in $\mathcal{F}$ has a subsequence that either (i) converges uniformly on every compact subset of U or (ii) converges uniformly to ∞ on U.

Of particular interest to us will be Montel's theorem which we state without proof.

Theorem **2.1.3** (Montel)

If $\cup_f \in \mathcal{F}[f(U)]$ omits two or more points in the complex plane, then f is a normal family on U.

Equivalently, if $\mathcal{F}$ is not normal on U, then $\cup_f \in \mathcal{F}[f(U)]$ omits at most one point in the complex plane.

In dealing with entire transcendental functions like λe^z, we have access to a particularly powerful tool. We recall that such functions have an essential singularity at ∞. Hence, they satisfy the hypothesis of the great Picard theorem.

Theorem **2.1.4**

Suppose an analytic function f has an essential singularity at $z = a$. Then for any neighborhood U of a and for all $z^* \in \mathbb{C}$ (with at most one exception) there exist infinitely many $z \in U$. such that $f(z) = z^*$. We can now give a formal definition of the Julia set. We consider $\mathcal{F} = \{f^n\}$ where f^n is the n^{th} iterate of f. We define the Julia set of f, $J(f)$, by $J(f) = \{z: \{f^n\} \text{ is not normal on any neighborhood of } z\}$.

Remark **2.1.1**

We remark here that the Julia set is completely invariant. That is if a the point is contained in $J(f)$, then so are all of its images and all its pre images. In addition, from this definition we may immediately conclude that all repelling period points are contained in $J(f)$.

In contrast to the above remark, the attracting fixed points and their basin of attraction are never contained in the Julia set. We note in addition that any point in the basin of attraction of a fixed point z_0 has about it a neighborhood U such that if $z \in U$, then $\lim_{n\to\infty} f^n(z) = z_0$. Hence f^n converges uniformly to the constant function $g(z) = z_0$ on U. We conclude that f^n is normal at z_0 and, therefore, $z_0 \notin J(f)$. Similar arguments give the same result for periodic points.

2.1.2 New perspectives in the dynamics of the exponential , $E_\lambda(z) = \lambda e^z, \lambda > 0$.

As a dynamical system on the real line, this map has two quite distinct dynamical behaviors depending upon weather $0 < \lambda < \frac{1}{e}$ or $\lambda > \frac{1}{e}$ the details of which have been discussed extensively elsewhere and would not be recounted here but refer the interested reader to Devaney and Durkin [7].

The change on the dynamics of E_λ which occurs at $\lambda = \frac{1}{e}$ is an example of a bifurcation known as a saddle-node bifurcation. Our interest will be to investigate the effects of this bifurcation in the complex plane. Before describing the Julia sets of E_λ, we recall some of the basic mapping properties of the exponential function.

1. $E_\lambda(x+iy) = \lambda e^{x+iy} = \lambda e^x(\cos x + i \sin y)$
2. Consequently, E_λ maps vertical lines $x = C$ to circles of radius λe^C centered at the origin, and horizontal lines $y = C$ to rays $\theta = C$ emanating from the origin. In particular, any rectangle with sides parallel to the axes and vertical length 2π is mapped into an annular region surrounding the origin.

Thus when $\lambda < \frac{1}{e}$ the Julia set $J(E_\lambda)$ is quite small as most points have orbits which are stable when $\lambda < \frac{1}{e}$. Recall that when $\lambda < \frac{1}{e}$ E_λ has two fixed points on $\mathbb{R}$: an attracting fixed point at q_λ and a repelling fixed point at p_λ.

Since

$$E_\lambda(1) = \lambda e \text{ and } E_\lambda(-\log\lambda) = 1,$$

we have

$$0 < q_\lambda < 1 < -\log\lambda < p_\lambda \qquad (2.1)$$

Consider the half plane

$$H = \{z: \text{Re } z < -\log\lambda\}$$

If $z \in H$ then

$$|E_\lambda'(z)| = |E_\lambda(z)| = \lambda \exp(\text{Re } z) < 1$$

Hence E_λ contracts H into the unit disk, which, since $-\log\lambda > 1$, is completely contained in H. Thus by the contraction mapping theorem, all points in H tend under iteration to a fixed point, which must be q_λ. Let us denote the basin of attraction of q_λ by $w(q_\lambda)$. Then $H \subset w(q_\lambda)$, and clearly, all points in $w(q_\lambda)$ have stable orbits. Since they all tend asymptotically to the same point.

As a consequence, we will be mainly interested in the complement of $w(q_\lambda)$. To investigate the dynamics of E_λ in the complement of $w(q_\lambda)$, we use (2.1) and choose ν such that

$$1 < E_\lambda(\nu) < -\log\lambda < \nu < p_\lambda \qquad (2.2)$$

Let H_ν denote the half plane $\text{Re } z < \nu$. Arguing as before, E_λ maps H_ν inside H and so H_ν is also contained in $W(q_\lambda)$.

Note that if $\text{Re } z > \nu$, then

$$|E_\lambda'(z)| = |E_\lambda(z)| = \lambda e^\nu > 1$$

which shows that E_λ is an expanding map on the complement of H_ν. Before discussing the properties of the complement of $w(q_\lambda)$ we first show that, for $\lambda < \frac{1}{e}$, this set is quite small. To do this we need an important Lemma that may be used severally in the sequel.

***Lemma* 2.1.2** (Expansion Lemma)

Suppose $|E_\lambda'(z)| > \mu$ for all $z \in B_\delta(z_0)$ where $\delta < \pi$. Then there is an open set $U \in B_\delta(z_0)$ such that $E_\lambda: U \to B_{\mu\delta}(E_\lambda(z_0))$ is a homeomorphism.

Proof

Assume the statement of the Lemma. Assume that $\delta < \pi$, then E_λ is one–to-one in $B_\delta(z_0)$. Hence we may define the inverse map $L: E_\lambda(B_\delta(z_0)) \to B_\delta(z_0)$. From the chain rule, it follows that $|L^{'}(z)| < \frac{1}{\mu}$ for all $z \in E_\lambda(B_\delta(z_0))$. Thus if $|z_0 - z_1| = \delta$, we have

$$\delta = |z_0 - z_1| = |L(E_\lambda(z_0)) - L(E_\lambda(z_1))| \le \frac{1}{\mu}|E_\lambda(z_0) - E_\lambda(z_1)|$$

It follows that E_λ maps $|z_0 - z_1| = \delta$ to a curve which contains $B_{\mu\delta}(E_\lambda(z_0))$ in its interior. By the maximum principle,

$$E_\lambda(B_\delta(z_0)) \supset B_{\mu\delta}(E_\lambda(z_0))$$

Thus we see that, as long as the orbit of a point remains in the complement of H_ν where

$$|E_\lambda^{'}(z)| = |E_\lambda(z)| \ge \nu > 1,$$

successive iterations of E_λ tend to expand neighborhoods of the original points, we may apply this idea repeatedly to prove the following result.

***Theorem* 2.1.5**

Suppose z_0 lies in the complement of $w(q_\lambda)$ and suppose U is an open set containing z_0. Then $U \cap w(q_\lambda) \neq \emptyset$

Proof

Suppose on the contrary that

$$E_\lambda^{\,n}(U) \cap w(q_\lambda) = \emptyset \text{ for all } n.$$

It follows that

$$E_\lambda^{\,n}(U) \cap H_\nu = \emptyset \text{ for all } n.$$

Hence we have

$$|E_\lambda^{'}(z)| > \mu > 1$$

For all $z \in U$, where $\mu = \lambda e^{\nu}$.Choose δ so that $B_\delta(z_0) \subset U$. Then by the expansion lemma

$$B_\lambda(B_\delta(z_0)) \supset B_{\mu\delta}(E_\lambda(z_0)).$$

Now $B_{\mu\delta}(E_\lambda(z_0))$ does not meet H_ν for otherwise we have a contradiction. It follows that we may apply repeatedly the lemma to this disk. Continuing in this fashion, and using the fact that $E_\lambda^{\,n}(z_0) \not\subset H_\nu$, it follows that we may find a disk of radius $\mu^n\delta$ about $E_\lambda^{\,n}(z_0)$ which does not meet H_ν. If we choose n large enough so that $\mu^n\delta > 2\pi$, then this disk must meet one of the horizontal lines of the form $y = (2k+1)\pi, k \in \mathbb{Z}$.

But these lines are mapped by E_λ onto the negative real axis which lies in H_ν. This contradiction establish the result.

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