Journal of the Nigerian Association of Mathematical Physics
***Volume* 15 (November, 2009), pp 9 - 12**

A note on connected simple graph of order $n \geq 4$

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Abstract

In this paper, we investigate the existence of a class of connected simple graph, G, with $|V_G| \geq 4$***, equipped with a special degree sequence, using Euler's Handshaking lemma* [5]. *We show that a graph of order,*** n***, with degree sequence*** $d(v_1) = n - 1$, $d(v_2) = n - 2$, … $d(v_{n-1}) = 1$, $d(v_n) = 1$, ***, does not exist at the moment*** $n \geq 4$. ***Furthermore we show that it does exist for smaller graphs and why this is so.***

Keywords
Connected graphs; simple graphs; Handshaking lemma; degree sequence

1.0 Introduction

Graph theory, basically, is the study of points, known here as vertices, and the interactions between these points. These interactions, which are commonly represented as lines joining these points, are called edges. (For rudimentary definitions in graph theory, the reader is referred to [1], [2], [3] and [4].) Given a vertex, v, say, of a simple graph, G, the degree of v is the number of other vertices of G with which v interacts directly. In other words, the number of direct links v has in G. When we say a graph G is connected, we are saying that every point of G can relate with one another directly or indirectly, and that G is simple if it neither has any loops nor parallel edges [4].

Euler in [5] showed that the sum of all the degrees of a graph G is twice the number of edges G contains. This has come to be known as the *Hand Shaking* lemma. In this paper, we show that there exist no such graph as G (connected, simple, of order $n \geq 4$) such that $d(v_1) = n - 1$, $d(v_2) = n - 2$, …, $d(v_{n-1}) = 1$, $d(v_n) = 1$. Recently works on connected graphs include [6] and [7]. In [6], some applications of connected graphs

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to quantum field theory; and in [7] an improved algorithm on generating connected graphs is presented. The degree sequence of a graph G of order n, is generally defined as the chain $d(v_1) \geq d(v_2) \geq, \ldots, \geq d(v_n)$ [4, Lemma 1.2]. However, in this work, we slightly modify this chain to $d(v_1) = n - 1$, $d(v_2) = n - 2$, …, $d(v_{n-1}) = 1$ and this is the first time such modification is been investigated

The paper is set as follows: Symbols used in the work are defined under Notations; the body of the work followed immediately after which a brief conclusion is given in form of an observation. The materials sited in the work are listed under References.

2.0 Notations

Most of the notations used in this paper are from [4]. G is a connected simple graph except otherwise defined. V_G is the vertex set of G, E_G is the edge set of G, $v_G = |V_G|$, i.e. the number of vertices of G. It is called the order G. $\varepsilon_G = |E_G|$is the number of edges in G, known as the size of G. $\Delta(G)$ and $\delta(G)$ are the maximal and minimal degrees of G respectively. $[0,n] = \{0,1,2,\ldots,n\}$, for any positive integer n which in general, implies that $[i,n] = \{i, i + 1, i + 2, \ldots, n\}$

3.0 Result

Here we present the results in this work.

***Lemma* 3.1**

Let G be a connected graph of order n. There exist at least two vertices v_x and v_y, $v_x, v_y \in V_G$, such that $d(v_x) = d(v_y)$.

Proof

Suppose that this claim is false and that $\exists$ a connected graph G' such that $\Delta(G') = d(v_1) > \ldots > d(v_n) = \delta(G')$, $\forall v_1, v_2, \ldots, v \in V(G')$. Let $d(v_1) \leq n - m$, $m \in [1, n - 1]$, (since G' is connected). Inductively, $d(v_2) \leq n - m - 1$, …, $d(v_{n-m}) \leq n - m - (n - m - 1) = 1$. If $d(v_{n-m}) = 0$, this implies that G' is disconnected, which is a contradiction. Likewise, if $d(v_{n-m}) = 1$, then, $d(v_{n-m-1}) = d(v_{n-m-2}) = \ldots = d(v_n) = 0$, , which implies still that G' is disconnected.

***Lemma* 3.1[5]:** For each graph G $\sum_{v \in V_G} d_G(v) = 2\varepsilon_G$.

Now we state the main result.

***Theorem* 3.1**

If G is said to be a connected graph of order n, $n \geq 4$, with a degree sequence such that $d(v_1) = n - 1$, $d(v_2) = n - 2, \ldots, d(v_{n-1}) = 1$, $d(v_n) = 1$. Then, G does not exist.

We should note that condition, $d(v_{n-1}) = 1 = d(v_n)$, is a direct implication of the claim in Lemma 3.1

***Proof of Theorem* 3.1**

Let G be a connected graph of order n, $n \geq 4$. It is not difficult to see that for any such natural number as n, $n = 4n' + n''$, $n' \in [1, m], n'' \in [0,3]$, for every integer m .

Claim 1

$\sum_{r=1}^{4n'-1} 4n' + r,$ is even (since $\frac{(4n'-1)(4n')}{2} = 2(4n'^2 - n')$,) and $\sum_{r=1}^{4n'} 4n' + 1 - r$ is even (since $\frac{4n'(4n'+1)}{2} = 2(4n'^2 - n')$).

Claim 2

Both $\sum_{r=1}^{4n'+1} 4n' + 2 - r,$ $\sum_{r=1}^{4n'+2} 4n' + 3 - r,$ are odd. (This is easy to see by following the procedure in Claim 1.)

Now, let H be a member of the class of n – ordered connected graph $\hat{G}$, for which $d(v_1) = n - 1$, $d(v_2) = n - 2, \ldots, d(v_{n-1}) = 1$, $d(v_n) = 1$ with $n = 4p + q'$ $p \in [1, m]$, $q' \in [0,1]$. We can see that $\sum_{v \in V_H} d_v(H)$ is odd, from Claim 1 above, which contradicts Lemma 1.1.

For $n = 4p + p''$, $q'' \in [2,3]$, let $\underline{H}$ be some arbitrary member of n – ordered graph $\hat{G}$. By claim II, $\sum_{v \in V_{\underline{H}}} d_v(\underline{H})$ is even, which conforms with Lemma 1.1. However, it is clear that $\check{G}$ is an extension of $\hat{G}$. Thus, $\check{G}$ does not exist. □

4.0 Conclusion

We have shown that graph G with the defined degree sequence does not exist when $n > 3$. However, if $n \leq 3$, we see that Theorem 3.1 does not holds for $n = 3$, since $d(v_1)$ can be equal to 2 while both $d(v_2)$ and $d(v_3)$ can still be 1. For $n = 2$, $d(v_1) = d(v_2)$ while the $n = 1$ case is trivial.

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