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Number of permutations with a given cycle structure

[1]Ali Bashir and [2]M. S. Audu
[1]Department of Mathematics,
Nigerian Defence Academy, Kaduna, Nigeria
[2]Department of Mathematics,
University of Jos, Jos, Nigeria

Abstract

Let X_n be an n-element set. We give an alternative proof of Cauchy's theorem for the number of permutations with a given cycle structure in X_n.

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1.0 Introduction and preliminaries

Let $X_n = \{a_1, a_2, \ldots, a_n\}$ be a finite set of arbitrary elements, a permutation on X_n is a one-to-one mapping of X_n onto itself. The set of all permutations on X_n forms a group with respect to permutation multiplication (composition of mappings) called the symmetric group of degree n, denoted as S_n *or* $Sym(X_n)$ ($|S_n| = n!$). A Permutation group G is a subgroup of a symmetric group.

The identity permutation on X_n is the identity mapping which leaves all points of X_n fixed, $i : x \to x$ $(x)i = x \ \forall x \in G$. Any element $g \in Sym(X_n)$ can be written in $r-$cycle i.e. $g = (x_1 \ x_2 \ \ldots \ x_r)$, such that x_1 is mapped to x_2, x_2 is mapped to x_3, … x_{r-1} and x_r is mapped to x_1 and any other element of X_n to itself. The length of a cycle is the number of distinct elements (points) which occur in the cycle.

Each cycle can be decomposed uniquely into disjoint cycles. A cycle which interchanges only two points and fixes the rest is called a transposition. Every permutation can be written as a product of transpositions, $g = (x_1 \ y_1)(x_2 \ y_2) \ldots (x_n \ y_n)$.

Recall from [1] that an even permutation is a permutation which can be expressed as a product of an even number of cycles of even length and/or a product of any number of cycles of odd length. A permutation that is not even is called odd. The set of even permutations of X_n, is called the alternating group and is usually denoted by A_n.

[1]Corresponding author:
[1]e-mail address: bashirklgi@yahoo.com

[1]Telephone Number: +234-08020591271, +234-08033745340

2.0 Number of cycle structure in $\alpha = S_n$.

Let $X_{mn} = \{a_1, a_2, \ldots, a_{mn}\}$, where $m \geq 2$ *and* $n \geq 1$, we easily obtained the following lemmas:

***Lemma* 2.1**

Let $X_n = \{a_1, a_2, \ldots, a_n\}$ The number of ways in which a permutation α of $X_4 = \{a_1, a_2, a_3, a_4\}$ can be expressed as a product of two transpositions is 3.

***Lemma* 2.2**

The number of ways in which a permutation α of X_6 can be expressed as a product of three transpositions is 15

***Lemma* 2.3**

The number of ways in which permutations α *of* X_{2n} can be expressed as a product of n transpositions is, $\dfrac{(2n)!}{2n \cdot 2(n-2) \ldots 4 \cdot 2} = \dfrac{(2n)!}{2^n \cdot n!}$.

***Lemma* 2.4**

Let $X_n = \{a_1, a_2, \ldots, a_n\}$ The number of ways in which a permutation α of $X_6 = \{a_1, a_2, a_3, a_4\}$ can be expressed as a product of two $3-$cycles is 40

***Lemma* 2.5**

The number of ways in which a permutation α of X_9 can be expressed as a product of three $3-$cycles is 1680

***Lemma* 2.6**

The number of ways in which permutations α *of* X_{2n} can be expressed as a product of n $3-$cycles is, $\binom{3n-1}{2}\binom{3n-3}{2} \ldots \binom{5}{2}\binom{2}{2}(3!)^n = \dfrac{(3n)!}{3^n \cdot n!}$.

***Lemma* 2.7**

The number of permutation α of X_{mr} that can be expressed as a product of r m-cycles is , $\dfrac{(mr)!}{m^r\, r!} = f(r, m)$.

Proof

If $r = 1$, we have one m-cycle on m elements of which there are clearly $(m-1)!$ m-cycles which agrees with our formula, $f(1, m) = \dfrac{m!}{m1!} = (m-1)!$. If $m = 2$ it reduces to Lemma 2.3

Now let the first m-cycles be $(ax_1x_2 \ldots x_{m-1})$. There we choose $x_1x_2 \ldots x_{m-1}$ in $\binom{mn-1}{m-1}$ way and there are $(m-1)!$ ways of writing this first m-cycle. The remaining $m(n-1)$ elements can be expressed as a product of m-cycles in $\dfrac{(m(n-1))!}{m^{n-1}(n-1)!}$

Thus we have : $= \frac{(mn-1)!\,(m-1)!}{(m-1)!\,(m(n-1))!}\frac{m(n-1)!}{m^{n-1}(n-1)!}\frac{mn}{mn} = \frac{(mn)!}{m^n n!}$

***Theorem* 2.8**

Let α be a permutations of X_n with r_i $(m-i+1)$-cycles $(i = 1,2,\ldots,m-1)$. Then the number of such permutations is $\frac{n!}{m^{r_1} r_1!(m-1)^{r_2} r_2!\ldots 2^{r_{m-1}} r_{m-1}!} = \frac{n!}{\prod_{i=0}^{m-2}(m-i)^{r_{i+1}}(r_{i+1})!}$

Proof

First note that $mr_1 + (m-1)r_2 + \ldots + 3r_{m-2} + 2r_{m-1} = n$. Now choose mr_1 elements from X_n to form r_1 $m-$cycles. This can be done in $\binom{n}{mr_1}$ ways, and these mr_1 elements can be expressed as a product of r_1 $m-$cycles in $f(r_1, m)$. ways Next choose $(m-1)r_2$ elements from the remaining $n-mr_1$ elements to form the r_2 $(m-1)-cycles$. This can be done in $\binom{n-mr_1}{(m-1)r_2}$ ways and these $(m-1)r_2$ elements can be expressed as a product of r_2 $(m-1)-cycles$, in $f(r_2, m-1)$ ways. We continue in this way until we reach the last $2\,r_{m-1}$ elements which can be expressed as a product of r_{m-1} $2-cycles\ in\ f(r_{m-1}, 2)$ ways. Multiplying all the possibilities gives

$$\binom{n}{mr_1} f(r_1, m).\binom{n-mr_1}{(m-1)r_2} f(r_2, m-1) \ldots \binom{2r_{m-1}}{2r_{m-1}} f(r_{m-1}, 2).$$

$$= \frac{n!}{(n-mr_1)!mr_1!} f(r_1, m)\frac{(n-mr_1)!}{(n-(r_1-r_2)m-r_2)!(m-1)r_2!} f(r_2, m-1) \ldots \frac{2r_{m-1}!}{0!2r_{m-1}!} f(r_{m-1}, 2).$$

$$= \frac{n!}{(n-mr_1)!mr_1!} f(r_1, m)\frac{(n-mr_1)!}{(n-(r_1-r_2)m-r_2)!(m-1)r_2!} f(r_2, m-1) \ldots$$

$$\binom{n-mr_1-(m-1)r_2-\ldots-3r_{m-2}}{2r_{m-1}} f(r_{m-1}, 2).$$

$$= \frac{n!}{(n-mr_1)!mr_1!}\frac{(mr_1)!}{m^{r_1}r_1!}\frac{(n-mr_1)!}{(n-(r_1-r_2)m-r_2)!(m-1)r_2!}\frac{[(m-1)r_2]!}{(m-1)^{r_2}r_2!} \ldots$$

$$\binom{n-mr_1-(m-1)r_2-\ldots-3r_{m-2}}{2r_{m-1}}\frac{(2r_{m-1})!}{2^{r_{mm-1}}(r_{m-1)!}}.$$

$$= \frac{n!}{m^{r_1} r_1!(m-1)^{r_2} r_2!\ldots 2^{r_{m-1}} r_{m-1}!}$$

This simplifies to the required result by using Lemma 2.4 and algebraic manipulations.

References

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