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On the group structure and root system of SL_n over a field

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Abstract

Given a commutative field F, the Whitehead functor K_1 and Steinberge functor K_2 are closely related to the theory of general linear group through exact sequences of groups. In this paper, the group structure of SL_n over a field F is closely examined and its root system is computed. Only the case n = 3 is considered.

Keywords
Special linear group, root system, cartan integer.

MSC: 17BXX, 17B20

1.0 Introduction

Let F be a field, we consider the groups of matrices

$$GL_n(\mathbb{F}) := \{(x_{rt}) \in M_n(\mathbb{F}) : 1 \le r, t \le n, \det(x_{rt}) \neq 0\}.$$
$$SL_n(\mathbb{F}) := \{(x_{rt}) \in GL_n(\mathbb{F}) \mid \det(x_{rt}) = 1\}.$$

It is known from [5], there is a special homomorphism, $i_n : GL_n(\mathbb{F}) \to GL_{n+1}(\mathbb{F})$ induced by the embedding of $\mathbb{F}^n$ in $\mathbb{F}^{n+1}$ z1, $(a_1, a_1, ..., a_n)$ α $(a_1, a_1, ..., a_n, 0)$ each i_n is a monomorphism. Identification of $GL_n(\mathbb{F})$ with its image $GL_{n+1}(\mathbb{F})$ under i_n gives the tower

$$GL_1(\mathbb{F}) \subset GL_2(\mathbb{F}) \subset GL_3(\mathbb{F}) \subset GL_4(\mathbb{F}) \subset ...$$

The tower group $GL(\mathbb{F})$ is given by

$$\mathrm{Y}_{n=1}^{\infty} GL_n(\mathbb{F}) = \lim_{n \to \infty} GL_n(\mathbb{F}) \qquad (*)$$

This result is also true for $SL_n(\mathbb{F})$. Here, $GL(\mathbb{F})$ and $SL(\mathbb{F})$ are inductive limits of the well known matrix groups of the general and special linear group over $\mathbb{F}$.

2.0 The group structure of SL_n over a field

Let F be any field and F^* denote the multiplicative group of F. The determinant map $\det : GL(\mathbb{F}) \to \mathbb{F}$ yields an exact sequence of groups.

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$$1 - SL(\mathbb{F}) \to GL(\mathbb{F}) \to \mathbb{F}^* \to 1.$$

Let $e_{ij} = (x_{tr})$ be the matrix with coefficients in $\mathbb{F}$ such that $x_{rt} = 1$ if $(r,t) = (i,j)$ and $x_{rt} = 0$ otherwise, and let $1 = 1_n \in GL_n(\mathbb{F})$ denote the identity matrix. For any $a \in \mathbb{F} i, j$ $= 1, \ldots, n \ \ i \neq j$, we define the matrices

$$r_{ij}(a) = 1_n + ae_{ij} \quad (a \in \mathbb{F})$$

$$s_{ij}(a) = r_{ij}(a) u_{ji}(-a^{-1}) r_{ij}(a) \quad (a \in \mathbb{F}, a \neq 0)$$

$$t_{ij}(a) = s_{ij}(a) s_{ij}(-1) \quad (a \in \mathbb{F}, a \neq 0)$$

The elements $r_{ij}(a), s_{ij}(a), t_{ij}(a)$ give the following relations:

(*i*) $r_{ij}(a+b) = r_{ij}(a) r_{ij}(b)$

(*ii*) $$\left[r_{ij}(a), r_{kl}(b)\right] = \begin{cases} r_{ij}(ab), \text{if } i \neq l, j = k \\ r_{jk}(-ab), \text{if } i = l, j \neq k \\ 1, \text{otherwise with condition that } (i,j) \neq (j,i) \end{cases}$$

(*iii*) $s_{ij}(a) r_{ij}(b) s_{ij}(a)^{-1} = r_{ij}(-a+b)$ for $a \in \mathbb{F}^*, \ b \in \mathbb{F}$

(*iv*) $t_{ij}(ab) = t_{ij}(a) t_{ij}(b)$

***Theorem* 2.1** [4]

(*a*) The group $SL_n(\mathbb{F})$ is generated by the matrices $\{r_{ij}(a) : 1 \leq i, j < n \ \ i \neq j \ \ a \in \mathbb{F}\}$

(b) The matrices $\{s_{ij}(a) : 1 \leq i, j \leq n \ \ i \neq j \ \ a \in \mathbb{F}^*\}$ generate the subgroup *M* of all monomial matrices of $SL_n(\mathbb{F})$.

(c) The matrices $\{t_{ij}(a) : 1 \leq i, j < n, i \neq j, a \in \mathbb{F}^*\}$ generate the subgroup *U* of all diagonal matrices of $SL_n(\mathbb{F})$.

***Remark* 2.1**

The subgroup *M* is the normalizer of *U* in $SL_n(\mathbb{F})$ and the quotient M/U is isomorphic to the symmetric group S_n.

***Theorem* 2.2** [10]

Let a presentation of *G* be given by relations (*i*), (*ii*) and (*iv*) if $n \geq 3$ and (*i*), (*iii*) and (iv) if $n = 2$. Let denote by $\hat{G}$ the group given by the presentation (*i*), (*ii*) if $n \geq 3$ and (*i*), (*iii*) if $n = 2$. Then the canonical map $\pi : \tilde{G} \to G$ is central. Assume $|k| > 4$ if $n \geq 3$ and $|k| \neq 4,9$ if $n = 2$. Then this central extension is universal i.e. every central extension $\pi_1 : G_1 \to G$ factors from π.

***Remark* 2.2**

As *G* and $\tilde{G}$ are perfect, $\tilde{G}$ as a universal extension of *G* is unique up to isomorphism. The group $St_n(\mathbb{F}) = \tilde{G}$ is called the Steinberg group of $SL_n(\mathbb{F})$.

3.0 Root systems of $SL_n(\mathbb{F})$

Definition 3.1

Let X be a finite dimensional $\mathbb{R}$-vector space with scalar product $\langle\ ,\ \rangle$. A set $\sum \subset X \setminus \{0\}$ is a root system in X if the following hold.

(*i*) The set $\sum$ is finite, generate *V*, and $-\sum = \sum$

(*ii*) For each $\alpha \in \sum$, the linear map

$S_\alpha : V \to V$ defined by

$$S_\alpha(r) = r - 2\frac{\langle \alpha, r\rangle}{\langle \alpha, \alpha\rangle}\alpha \text{ leaves}$$

$\sum$ invariant : $S_\alpha\left(\sum\right) = \sum$.

(iii) For each pair $\alpha, \beta \in \sum$, the number

$$n_{\beta,\alpha} = 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \text{ is an integer.}$$

This is called "Cartan integer".

***Definition* 3.2**

$\sum$ is reducible if there exist proper mutually orthogonal sub-spaces X', X'' of *X* such that $X = X' \perp X''$ and $\sum = \left(X' \cap \sum\right) \cup \left(X'' \cap \sum\right)$. Otherwise $\sum$ is called irreducible.

***Definition* 3.3**

An element in $\sum$ is called a simple if it is not the sum of two positive roots.

***Proposition* 3.4**

Every root is an integral sum of simple roots with coefficients of same sign.

Next we give the main result.

4.0 Computation of the root system of $SL_3(\mathbb{F})$

Let $Diag_3(\mathbb{F})$ denote the subgroup of all diagonal matrices in $GL_3(\mathbb{F})$, and we denote a diagonal matrix just by its components: we define

$$diag(dr) = diag(d_1, d_2, d_3) := \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in Diag_3$$

The subgroup *U* in Theorem 2.1(c) is strictly contained in $Diag_3$ and as a result we define homomorphism $\Gamma_1, \Gamma_2, \Gamma_3$ respectively by

$$\Gamma_1 : Diag_3(\mathbb{F}) \to \mathbb{F}^*$$

$$\Gamma_1(diag(d_r)) = d_1$$

$$\Gamma_2 : Diag_3(\mathbb{F}) \to \mathbb{F}^*$$

$$\Gamma_2(diag(d_r)) = d_2$$

$$\Gamma_3 : Diag_3(\mathbb{F}) \to \mathbb{F}^*$$
$$\Gamma_3(diag(d_r)) = d_3$$

Therefore the set $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ is a basis of the free $\mathbb{Z}$-module $Y(Diag_3)$ of all homomorphisms

$$\chi : Diag_3 \to \mathbb{F}^*$$

Hence the set $\{\alpha_i = \Gamma_i - \Gamma_{i+1} : \; i = 1,2\}$ is a basis of the $\mathbb{Z}$-module $Y(U)$ of all homomorphism on *U*. From this basis we obtain the following

$$\langle \alpha_1, \alpha_1 \rangle = 2 \qquad \langle \alpha_2, \alpha_2 \rangle = 2$$
$$\langle \alpha_1, \alpha_2 \rangle = -1 \qquad \frac{\langle \alpha_1, \alpha_2 \rangle}{2} = \frac{-1}{2}$$
$$\langle \alpha_2, \alpha_1 \rangle = -1 \qquad \langle \alpha_2, \alpha_1 \rangle = -\tfrac{1}{2}$$

The root system is given by $\sum = \{\alpha_{ij} = \Gamma_i - \Gamma_j \,|\, 1 \le i, j \le n \;\; i \ne j\}$ $\left(\text{hence } \alpha_{i,i+1} = \alpha_i\right)$. Therefore

$$\alpha_{ij} = \begin{cases} \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} & \text{if } \; i < j \\ -\alpha - \alpha_{j+1} - \ldots - \alpha_{j-1} & \text{if } \; i > j \end{cases}$$

i.e. every element in $\sum$ is an integral combination of d_i coefficients of the same sign. Therefore the root system $\sum$ is given by $\{\pm\alpha_1 \;\; \pm\alpha_2, \;\; \pm(\alpha_1 + \alpha_2) = \pm\alpha_3\}$. In general every pair (i, j) of indices with $i \ne j$ in $SL_n(\mathbb{F})$ determines a root $\alpha_{ij} = \Gamma_i - \Gamma_j \in \sum$.

5.0 Conclusion

In this paper, we compute a special case of a well motivated problem. Further work is in progress to generalize these results using recent development in enumerative geometry.

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