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Periodic solutions for a boundary value problem of a third order ordinary differential equation

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Abstract

Existence results for some non-linear ordinary differential equations* (1.1) – (1.2*) have been very difficult to establish when it comes to computation of the apriori bounds. These difficulties were due to the nature of Lyapunov functions involved. In this paper, these difficulties have been avoided by the use of integrated equation as the mode of estimating the apriori bounds.

Keywords
Nonlinear ODE, Boundary value problems, Integrated equations, apriori bounds.

1.0 Introduction and formulation of Theorem 1.1

Consider the third order non-linear boundary value problem

$$\dddot{x} + f(\ddot{x}) + g(\ddot{x}) + h(x) = p(t) \tag{1.1}$$

with boundary conditions

$$D^{(r)}x(0) = D^{(r)}x(2\pi), \text{ r = 0, 1, 2, } D = \frac{d}{dt} \tag{1.2}$$

where *f, g, h* and *p* are continuous functions depending on the arguments shown and *p* is *2π* periodic in t. We note that equation (1.1) is the most general form of the constant coefficient equation.

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = p(t) \tag{1.3}$$

in which *a, b, c,* are constants and *p* is a continuous function and *2π* periodic in *t*. It is well known that if the Routh-Hurwitz's conditions.

$$a > 0, b > 0, ab > c > 0 \tag{1.4}$$

hold, the roots of the auxiliary equation

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0 \tag{1.5}$$

have negative real parts. Then existence of periodic solutions when *p* is also *2π* periodic in *t* can be verified for (1.3), when (1.4) holds.

Extensions of equation (1.3) to its non-linear terms where *a, b, c,* are all not necessarily constants are available in the literature. For instance, Ezeilo [2] proved the existence of at least one harmonic oscillation for the equation

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$$\dddot{x}+a\ddot{x}+b\dot{x}+h(x)=p(t) \quad (1.6)$$

where h is continuous. A similar result has also been proved by Pliss [8] for the equation

$$\dddot{x}+a\ddot{x}+b\dot{x}+h(x)=p(t,x,\dot{x},\ddot{x}) \quad (1.7)$$

in which *h(x)* satisfies a slightly improved generalized Routh-Hurwitz's conditions and the forcing term p now depends on $t,x,\dot{x},\ddot{x}$ and it is bounded for all values of its arguments. Reissig, Sansone and Conti [9] proved an existence result for the equation

$$\dddot{x}+\varphi(\ddot{x})+b\dot{x}+cx=p(t) \quad (1.8)$$

when $b>0$, $c>0$, $|p(t)|\le m$, $m>0$ for $t>0$. Tejumola [11] proved the existence of periodic solutions for the equation

$$\dddot{x}+f(\dot{x})\ddot{x}+g(\dot{x})+h(x)=p(t,x,\dot{x},\ddot{x}) \quad (1.9)$$

in which *f, g, h* and *p* are continuous functions depending on the arguments shown. Very recently, Tejumola [13] proved existence of no prontrivial periodic solutions for the equation

$$\dddot{x}+h_1(\ddot{x})+h_2(\dot{x})+c_3x=p(t,x,\dot{x},\ddot{x}) \quad (1.10)$$

Most of these results depended on the availability of a suitable boundedness result using the well-known Routh-Hurwitz's conditions. Further results on the extension of equation (1.3) to its non-linear terms are available in Ezeilo [2, 3, 4, 5, 6], Tejumola [11, 12] and still more can be found in Reissig, Sansone and Conti [9]. However, Villari [14] proved an existence result without the use of generalized Routh-Hurwitz's criteria for the equation.

$$\dddot{x}+\varphi(\ddot{x})+b_1(\dot{x})+c_1(x)-p(t),\ \ b_1<0, c_1>0 \quad (1.11)$$

subject to the condition $\{\varphi(z_1)-\varphi(z_2)\}(z_1-z_2)<0$ for $z_1\neq z_2$ where $z_1=\dot{x}_1$, $z_2=\ddot{x}_1$ and used Halanay's approach in his proof. Ezeilo [4] proved the existence of periodic solutions for the equation (1.7) and has shown that the condition

$$0<x^{-1}h(x)<ab,\ \ |x|\ge \mathbf{R} \quad (1.12)$$

or other similar conditions are not absolute necessity for the existence of periodic solutions for equation (1.7). This can be seen clearly from the consideration of the linear equation

$$\dddot{x}+\ddot{x}+4\dot{x}+30x=\cos t$$

which has periodic solution, *850x = 29cost – 3sint* but the corresponding *a, b* and *h* in (1.7) do not satisfy (1.12).

The objective of this paper is to give some other results in the "non-Routh Hurwitz's" direction. That is, $a>0$, $b>0$, $ab<c$ (1.13)
or other similar conditions for equation (1.1). Hence, we propose a theorem whereby the "non-Routh Hurwitz's" conditions could be generalized to equation (1.1). By comparison of equation (1.1) with (1.3), we observe that equation (1.1) is equivalent to (1.3) if

$$\left.\begin{array}{l} f(\ddot{x}) \text{ is replaced by } a\ddot{x} \\ g(x) \text{ is replaced by } b\dot{x} \\ \text{and} \\ h(x) \text{ is replaced by } cx \end{array}\right\} \quad (1.14)$$

This may in turn suggest $\ddot{x}f(\ddot{x})$ and $h'(x)$ being replaced by $a\ddot{x}$ and c respectively. To be more precise, let us take the purely imaginary root

$$\lambda=i\beta,\ \beta>0 \text{ if } a\neq 0 \text{ and } a^{-1}c\ \neq\ \beta^2 \quad (1.15)$$

$(\beta$ an integer$)$. Thus if P is 2π periodic in t, the linear differential equation (1.3) has indeed

2π periodic solutions if *a, b, c* are subject to condition (1.15). Thus we have the following in which the hypotheses have been suggested by equation (1.14).

***Theorem* 1.1:**

Suppose in addition to the basic assumption on *f, g, h* and *p*, there exists constants $a > 0, c > 0$ and $\beta > 0$ such that the function $h(x)$ satisfies

$$h'(x) \le c < a \quad \forall \text{ x} \tag{1.16}$$

$$a - c > 0,\ a^{-1}c \ne \beta^2,\ a \ne 0 \tag{1.17}$$

$$zf(z) \le az^2 + Bz \qquad 1.18)$$

$$|h(x)| \to +\infty \text{ as } |\text{x}| \to \infty \tag{1.19}$$

The function *p* is bounded and 2π periodic in *t*. There is one 2π periodic solution in $L'[0, 2\pi]$ for arbitrary $g(y)$.

2.0 Notations

Throughout the proof, which follows, we denote finite capitals C_1, C_2, C_3, … which depend at most on *f, g, h* and *p*. The C_{ij} $(i = 0, 1, 2, \ldots)$ retain a fixed identity throughout the proof of theorem 1.1.

The symbols $|\cdot|_\infty$, $|\cdot|_1$, $|\cdot|_2$ with respect to the mappings: $[0,\ 2\pi] \to \mathbb{R}$ will have their usual meaning. That is for a given function $\theta:\ [0,\ 2\pi] \to \mathbb{R}$ say

$$|\theta|_\infty \ := \max_{0 \le t \le 2\theta} |\theta|,\ \ |\theta|_1 \ := \int_0^{2\pi} |\theta(s)| ds,\ \ |\theta|_2 \ := \left(\int_0^{2\pi} \theta^2(s) ds \right)^{\frac{1}{2}}$$

3.0 Proof of Theorem 1.1

The proof of theorem 1 shall be by the Leray-Schauder fixed point technique (see Leray and Schauder [7]) and instead of equation (1.1), we consider the parameter λ-dependent equation

$$\dddot{x} + f_\lambda(\ddot{x}) + \lambda g(\dot{x}) + h_\lambda(x) = \lambda p \tag{3.1}$$

where

$$f_\lambda(\ddot{x}) = (1-\lambda)a\ddot{x} + \lambda f(\ddot{x}) \tag{3.2}$$

$$h_\lambda(x) = (1-\lambda)cx + \lambda h(x) \tag{3.3}$$

The equation (3.1) can be written in matrix form by setting

$$\dot{x} = y,\ \dot{\text{y}} = z,\ \dot{\text{z}} = -f_\lambda(\ddot{x}) - \lambda g(\dot{x}) - h_\lambda(x) + \lambda p \tag{3.4}$$

and equation (3.4) can be written compactly in the form

$$\dot{X} = AX + \lambda F(X, t) \tag{3.5}$$

where

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \text{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & 0 & -a \end{pmatrix},\ \text{F} = \begin{pmatrix} 0 \\ 0 \\ Q \end{pmatrix} \tag{3.6}$$

with $Q = p - h(x) + cx - g(\dot{x}) - f(\ddot{x}) + a\ddot{x}$

We remark that equation (3.1) reduces to a linear equation

$$\dddot{x} + a\ddot{x} + cx = 0 \tag{3.7}$$

when $\lambda = 0$ and to equation (1.1) when $\lambda = 1$.

The auxiliary equation to (3.7)

$$\lambda^3 + a\lambda^2 + c = 0 \tag{3.8}$$

has no purely imaginary root $\lambda = i\beta, \beta > 0$ if $a \neq 0$ *and* $a^{-1}c \neq \beta^2$

Therefore the matrix ($e^{-2\pi A} - I$) (I being identity 3 x 3 matrix) is invertible. Thus $X = X(t)$ is a 2π periodic solution of (3.5) if and only if X satisfies the equation

$$X = \lambda TX.,\ 0 \leq \lambda \leq 1 \tag{3.9}$$

where the transformation T is defined by

$$\left(TX\right)(t) = \int_t^{t+2\pi} (e^{-2\pi A} - I)^{-1} e^{A(t-s)} F\left(X(s), s\right) ds \tag{3.10}$$

Let S be the space of all real-valued continuous and 3-vector function $\bar{X}(t) - x(t)$, y(t), z(t) which are of period 2π and with norm $\left\|\bar{X}\right\|_s$ defined by

$$\left\|\bar{X}\right\|_s = \sup_{0\leq t\leq 2\pi} \left\{|x(t)| + |y(t)| + |z(t)|\right\} \tag{3.11}$$

The definition of $\left(TX\right)(t)$ is the solution of the differentiated equation (3.5). Note that the matrices $\left(e^{-2\pi A} - I\right)^{-1}$ and $e^{A(t-s)}$ exist. Moreover $\left\|e^{A(t-s)}\right\|$ denotes the sum of absolute values of the elements of the matrix $e^{A(t-s)}$. Similarly $F(X, t)$ is continuous in X and t since f, g, h and p are continuous and is periodic in t with period 2π (by the periodicity of p). Since $\left(T.X\right)(t) = \int_t^{t+2\pi A} \left(e^{-2\pi} - I\right)^{-1} e^{A(t-s)} + \left(X(s), s\right) ds.$ Then a change of the variable $u = t - s$ in the integral equation yields

$$\left(TX\right)(t) = \int_0^{2\pi} \left(e^{-2\pi A} - I\right)^{-1} e^{Au} F\left(X(t-u),\ (t-u)\right) du \tag{3.12}$$

Also, let x(t) be a possible 2π periodic solution of the equation (1.1). By this assumption $F(X(t - u), (t - u))$ is equation (3.12) is definitely 2π periodic in u because of its composition as given by Q in equation (3.6). The term p is 2π periodic in t, the other terms $h(x)$, cx, $g(\dot{x})$, $f(\ddot{x})$ and $a\ddot{x}$ are all 2π periodic in t, by the assumption that $x(t)$ is a possible 2π periodic solution of equation (1.1). The 2π periodicity of $F(X(t), t)$ implies that $(TX)(t)$ is also 2π periodic in t. Thus $T : S \to S$.

Now $X(t) \in S$ and $X(t) = TX(t)$ implies that $X(t)$ is a 2π periodic solution of the differential equation $\dot{X} = AX + F\left(X,\ t\right)$. But

$$X(t) = \int_t^{t+2\pi} \left(e^{-2\pi A} - I\right)^{-1} e^{A(t-s)} F\left(X(s), s\right) ds$$

Differentiating both sides of the above integral equation with respect to t yields

$$\dot{X} = \left((e^{2\pi A} - I)^{-1}\right) A \int_t^{t+2\pi} e^{A(t-s)} F\left(X(s), s\right) ds + \left(e^{-2\pi A} - I\right)^{-1} e^{-2\pi A} F\left(X(t+2\pi), t+2\pi\right)$$

$$- \left(e^{-2\pi A} - I\right)^{-1} I \cdot F\left(X(t), t\right)$$

$$= A \int_t^{t+2\pi} \left(e^{-2\pi A} - I\right)^{-1} e^{A(t-s)} F\left(X(s), s\right) ds + \left(e^{-2\pi A} - I\right)\left(e^{-2\pi A} - I\right) F\left(X(t), t\right)$$

$$= AX + F\left(X(t), t\right)$$

That is $\dot{X} = AX + F\left(X(t), t\right)$. Again, consider the parameter λ differential equation (3.1) with the solution $X(t)=\lambda\int_t^{t+2\pi}\left(e^{-2\pi A}-I\right)^1 e^{A(t-s)}F\left(X(s),s\right)ds$ which by equation (3.10) is $X(t)=\lambda TX(t)$. Thus verifying the claim (3.9). It is therefore clear that the existence of a periodic solution of equation (3.5) for each $\lambda\in[0,1]$ would correspond to the existence of $X\in s$ satisfying (3.9). Thus the existence of at least *2π* periodic solution of equation (1.1) requires that there are constants C_6, C_3, C_9 independent of $\lambda\in[0,1]$ such that any *2π* periodic solution *X(t)* of equation (3.1) satisfies

$$|x|_\infty\le C_6,\ |\dot{x}|_\infty\le C_3\ and\ |\ddot{x}|_\infty\le C_9 \qquad (3.13)$$

see Scheafer [10].

Let *x(t)* be a possible *2π* periodic solution of equation (3.1). The main tool to be used here in this verification is the function *W* defined by

$$W=\tfrac{1}{2}\ddot{x}^2+\lambda G(\dot{x})+\dot{x}h_\lambda(x) \qquad (3.14)$$

where $G(\dot{x})=\int_0^{\dot{x}}g(s)ds$ and $h_\lambda(x)=(1-\lambda)cx+\lambda h(x)$. The time derivative $\dot{W}$ along the solution paths (3.4) is

$$\dot{W}=-\ddot{x}f_\lambda(\ddot{x})+\dot{x}^2h_\lambda'(x)+\lambda p\ddot{x} \qquad (3.15)$$

Integrating (3.15) with respect to *t* from *t = 0* to *t = 2π*, we have

$$W\Big|_0^{2\pi}=\int_0^{2\pi}\ddot{x}f_\lambda(\ddot{x})dt-\int_0^{2\pi}\dot{x}^2h_\lambda'(x)dt+\int_0^{2\pi}\lambda p\ddot{x}dt$$

By equation (3.2), (3.3), and (1.16), we have

$$0=-\int_0^{2\pi}(1-\lambda)a\ddot{x}^2dt-\int_0^{2\pi}\lambda f(\ddot{x})\ddot{x}dt+\int_0^{2\pi}(1-\lambda)c\dot{x}^2dt+\int_0^{2\pi}c\dot{x}^2dt+\int_0^{2\pi}\lambda p\ddot{x}dt$$

$$=-\int_0^{2\pi}(1-\lambda)a\ddot{x}^2dt-\int_0^{2\pi}\lambda f(\ddot{x})\ddot{x}dt+\int_0^{2\pi}c\dot{x}^2dt+\int_0^{2\pi}\lambda p\ddot{x}dt$$

By equation (1.18)

$$0\le-\int_0^{2\pi}(1-\lambda)a\ddot{x}^2dt-\int_0^{2\pi}\lambda a\ddot{x}^2dt-\int_0^{2\pi}B\ddot{x}dt+\int_0^{2\pi}c\dot{x}^2dt+\int_0^{2\pi}\lambda p\ddot{x}dt$$

Using the *2π* periodicity of *x(t)*, then

$$0\le-\int_0^{2\pi}(1-\lambda)a\ddot{x}^2dt-\int_0^{2\pi}\lambda a\ddot{x}^2dt+\int_0^{2\pi}c\dot{x}^2dt+\int_0^{2\pi}\lambda p\ddot{x}dt$$

After due simplification, we have, $0\le-\int_0^{2\pi}a\ddot{x}^2dt+\int_0^{2\pi}c\dot{x}^2dt+\int_0^{2\pi}\lambda p\ddot{x}dt$

or

$$\int_0^{2\pi}a\ddot{x}^2dt-\int_0^{2\pi}c\dot{x}^2dt\le\int_0^{2\pi}|p||\ddot{x}|dt \qquad (3.16)$$

$$\int_0^{2\pi}a\ddot{x}^2dt-\int_0^{2\pi}c\dot{x}^2dt\le M\int_0^{2\pi}|\ddot{x}|dt$$

We have used the bounded of *p* and the fact that *0 ≤ λ ≤ 1*. By the Fourier series expansion of *x(t)* $x(t)\approx a_0+\sum_{r=1}^{\infty}\left(ar\cos rt+br\sin rt\right)$ and the derivatives $\dot{x}(t)$ and $\ddot{x}(t)$, we got $\int_0^{2\pi}\ddot{x}^2dt\ge\int_0^{2\pi}\dot{x}^2dt$. Therefore by (3.16), $(a-c)\int_0^{2\pi}\ddot{x}^2dt\le M\int_0^{2\pi}|\ddot{x}|dt$. That is, $\int_0^{2\pi}\ddot{x}^2dt\le c_1\int_0^{2\pi}|\ddot{x}|dt$

$$\int_0^{2\pi}\ddot{x}^2dt\le c_1(2\pi)^{\frac{1}{2}}\left(\int_0^{2\pi}\ddot{x}^2dt\right)$$

By Schwartz's inequality, therefore

$$\left(\int_0^{2\pi}\ddot{x}^2dt\right)^{1/2}\leq c_1(2\pi)^{\frac{1}{2}}\equiv c_2 \tag{3.17}$$

Now since *x(0) = x(2π)*, there exists $\tau\in[0,\ 2\pi]$ such that $\dot{x}(\tau)=0$. So that the identity

$$\dot{x}(t)=\dot{x}(\tau)+\int_\tau^t\ddot{x}(s)ds$$

Therefore

$$\max_{0\leq t\leq 2\pi}\left|\dot{x}(t)\right|\leq\int_0^{2\pi}\left|\ddot{x}(t)\right|dt$$

$$\leq(2\pi)^{1/2}\left(\int_0^{2\pi}\ddot{x}^2ds\right)^{1/2}$$

By Schwartz's inequality. From (3.17) $\left|\dot{x}\right|_\infty\leq(2\pi)^{\frac{1}{2}}C_2=C_3$ (3.18)

Now integrating equation (3.1) with respect to t from $t = 0$ to $t = 2\pi$

$$\int_0^{2\pi}\dddot{x}dt+\int_0^{2\pi}f_\lambda(\ddot{x})dt+\int_0^{2\pi}\lambda g(\dot{x})dt+\int_0^{2\pi}\lambda_\lambda dt=\int_0^{2\pi}\lambda pdt$$

By equation (1.2), we have $\int_0^{2\pi}f_\lambda(\ddot{x})dt+\int_0^{2\pi}\lambda g(\dot{x})dt+\int_0^{2\pi}h_\lambda(x)dt=\int_0^{2\pi}\lambda pdt$

By equation (3.2) and (3.3), the above equation yields

$$\int_0^{2\pi}C_5\ddot{x}dt+\int_0^{2\pi}\lambda g(\dot{x})dt+\int_0^{2\pi}a_3xdt=\int_0^{2\pi}\left|\lambda\right|\left|p\right|dt$$

which yields

$$\int_0^{2\pi}(1-\lambda)a\ddot{x}dt+\int_0^{2\pi}\lambda f(\ddot{x})dt+\int_0^{2\pi}\lambda g(\dot{x})dt+\int_0^{2\pi}(1-\lambda)cxdt+\int_0^{2\pi}\lambda h(x)dt=\int_0^{2\pi}\lambda pdt$$

By (1.19), we have

$$\int_0^{2\pi}(1-\lambda)a\ddot{x}dt+\int_0^{2\pi}\lambda a\ddot{x}dt+\int_0^{2\pi}\lambda Bdt+\int_0^{2\pi}\lambda g(\dot{x})dt+\int_0^{2\pi}(1-\lambda)cxdt+\int_0^{2\pi}\lambda h(x)dt\leq\int_0^{2\pi}\left|\lambda p\right|dt$$ That is

$$\int_0^{2\pi}\lambda h(x)dt+\int_0^{2\pi}(1-\lambda)cxdt\leq\int_0^{2\pi}\left|\{\lambda p-\lambda g(\dot{x})\}\right|dt \tag{3.19}$$

By (3.18) and the boundedness of p combined with the fact that $0\leq\lambda\leq 1$ the right hand side of equation (3.19) is boundeded, so that $\int_0^{2\pi}\{(1-\lambda)cx+\lambda h(x)\}dt\leq C_4$.

That is,
$$\left|\int_0^{2\pi}\lambda p-\lambda g(\dot{x})dt\right|\leq C_4 \tag{3.20}$$

Which also implies that
$$\left|\int_0^{2\pi}(1-\lambda)cxdt+\int\lambda h(x)dt\right|\leq C_4 \tag{3.21}$$

But (3.21) implies

Given α>0 ∃ β>0 such that $\left|x\right|\geq\beta\Rightarrow\left|h(x)\right|>\alpha$ (3.22)

Then there exists $\tau\in\left[0,\ 2\pi\right]$ such that $\left|x(\tau)\right|\leq C_5$ (3.23)

Now if (i) $x(\tau)=0$, then we are done.

Suppose NOT i.e. (ii) $x(\tau)\neq 0$ for any τ, then the left hand side (3.19)

$$\left|\int_0^{2\pi}(1-\lambda)c\left|x\right|dt+\int\lambda\left|h(x)\right|dt\right|>\left|\int_0^{2\pi}(1-\lambda)c\beta dt+\int\lambda\alpha dt\right|$$

$$>\int_0^{2\pi}2\pi(1-\lambda)C\beta dt+2\pi\lambda\alpha$$

which implies that the left hand side of equation (3.19) is not bounded. This is a negation to the boundedness in equation (3.21). Therefore, equation (3.23) holds for $\tau \in [0,\ 2\pi]$. Thus, the identity $x(t) = x(\tau) + \int_{\tau}^{t} \dot{x}(t)dx$ holds. That is

$$\max_{0\le t\le 2\pi} |x(t)| \le |x(\tau)| + \int_0^{2\pi} \dot{x}(t)dt$$

$$\le C_5 + (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \dot{x}^2 dt \right)^{\frac{1}{2}}$$

By Schwartz's inequality from (3.18), $\max_{0\le t\le 2\pi} |x(t)| \le C_5 + C_3 \equiv C_6$.

Thus $$|x|_{\infty} \le C_6 \qquad (3.24)$$

It remains the third inequality (3.13) for the full realization of the proof of theorem 1.1. So consider equation (3.1) in the form $$\dddot{x} + f_{\lambda}(\ddot{x}) = K \qquad (3.25)$$

with $K = \lambda p - \lambda g(\dot{x}) - h_{\lambda}(x)$

In view of (3.18) and (3.24) combined with the boundedness of *p*, we are assumed that the right hand side of (3.25) is bounded.

That is $$K \le C_7 \qquad (3.26)$$

Now multiplying equation (3.25) by $\dddot{x}$ and integrate with respect to *t* from *t = 0* to *t = 2π*

$$\int_0^{2\pi} \dddot{x}^2 dt + \int_0^{2\pi} f_{\lambda}(\ddot{x})\dddot{x} = \int_0^{2\pi} K\dddot{x}dt \qquad (3.27)$$

By equations (3.2), (1.19) and (1.2), we have after due simplification that (3.27) reduces to

$$\int_0^{2\pi} \dddot{x}dt \le |K| \int_0^{2\pi} |\dddot{x}|dt$$

That is

$$\int_0^{2\pi} \dddot{x}^2 dt \le C_7 \int_0^{2\pi} |\dddot{x}|dt$$

$$\le C_7 (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \dddot{x}^2 dt \right)^{\frac{1}{2}}$$

By Schwartz's inequality. Therefore,

$$\int_0^{2\pi} \left(\dddot{x}^2 dt \right)^{\frac{1}{2}} \le C_7 (2\pi)^{\frac{1}{2}} \equiv C_8 \qquad (3.28)$$

Since $\dot{x}(0) = \dot{x}(2\pi)$ by (1.2) then there exists $\tau \in [0,\ 2\pi]$ such that $\ddot{x}(\tau) = 0.$ The identity

$$\ddot{x}(t) = \ddot{x}(\tau) + \int_{\tau}^{t} \dddot{x}(s)ds$$

holds. Therefore

$$\max_{0\le t\le 2\pi} |\ddot{x}(t)| \le \int_0^{2\pi} |\dddot{x}|\, dt$$

$$\le (2\pi)^{\frac{1}{2}} \left(\int_0^{2\pi} \dddot{x}^2 dt \right)^{\frac{1}{2}}$$

By Schwartz's inequality. From (3.28), $\max_{0\le t\le 2\pi} |\ddot{x}(t)| \le (2\pi)^{\frac{1}{2}} C_8 \equiv C_9$

Thus $$|\ddot{x}|_\infty \le C_9 \qquad (3.29)$$

4.0 Conclusion

Our estimates (3.18), (3.24), (3.29) verify equation (3.13) and the proof of theorem 1.1 follows, which implies existence of 2π periodic solutions for equation (1.1) subject to the boundary condition (1.2).

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