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# Periodic solutions of a certain nonlinear boundary value problem (BVP) of a fourth order differential equation

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***Abstract***

***The consideration of the eigenvalue approach and a comparison between the linear and nonlinear fourth order differential equation formed the basis for a theorem for existence of periodic solutions for the nonlinear boundary value problem of a fourth order differential equation. The proof of the theorem is by the Leray-Schauder fixed point technique with the use of integrated equation as the mode for estimating the a priori bounds.***

***Keywords***
Nonlinear ODE, Boundary value problem (BVP), a priori bounds, integrated equation, Leray-Schauder fixed point technique, parameter dependent equation.

## 1.0 Introduction and formulation of Theorem 1.1

Consider the eigenvalue problem

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_4 x = -a_1\lambda\dot{x} \tag{1.1}$$

with boundary conditions

$$D^{(r)}x(0) = D^{(r)}x(2\pi),\ r = 0,1,2,3,\ D = \frac{d}{dt} \tag{1.2}$$

where $a_1 \neq 0,\ a_2$ and $a_4$ are constants. Let $x(t)$ have the Fourier expansion

$$x'(t)\ -\ \sum_{r=1}^{\infty}\left(d_r\cos rt + e_r\sin rt\right) \text{ for } t\in\left[0,\ 2\pi\right]$$

split $x'(t)$ into two parts $\mathrm{Y}_1,\ \mathrm{Y}_2$ as follows

$$Y_1 = \sum_{r=\lambda}\left(d_r\cos rt + e_r\sin rt\right),\ Y_2 = \sum_{r>\lambda}\left(d_r\cos rt + e_r\sin rt\right)$$

Multiplying (1) by $Y_1 - Y_2$ and integrating over $\left[0,\ 2\pi\right]$, it can be checked that

(i) Any $\lambda \neq m^2$ for $m = 1,\ 2,\ \ldots$ is not an eigenvalue of (1.1) for arbitrary $a_2$ if $a_4 \neq 0$.

(ii) Any $\lambda = m^2$ for some $m = 1,\ 2,\ \ldots$ is an eigenvalue of (1.1) if and only if

$$\chi(m) = m^4 - a_2 m^2 + a_4 = 0.$$

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The statements ($i$) and ($ii$) are essential in the solvability of the well known 2π periodic BVP for the non autonomous equation.

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_1\lambda\dot{x} + a_4 x = P(t) \tag{1.3}$$

By ($i$) one expects existence of a 2π periodic solution for the equation

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + g(t,\dot{x}) + a_4 x = P(t) \tag{1.4}$$

for arbitrary $a_2,\ a_4$ where $a_1^{-1}g(t,y)$ is in the interval $\{m^2,\ (m+1)^2\}$ for $m$ a non zero integer Ezeilo [1] and Ezeilo and Onyia [3]. Again ($i$) taken together with ($ii$) leads one to expect periodic solutions to the 2π periodic BVP for the equation.

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = P(t,x,\dot{x},\ddot{x},\dddot{x}) \tag{1.5}$$

for arbitrary $a_1$ and $a_3$ if $a_2$ and $a_4$ satisfy

$$\chi(m) \neq 0 \text{ for } m = 1,\ 2\ ,... \tag{1.6}$$

The equation (1.6) is an improvement on the result of Ezeilo and Tejumola [2, 4] which required that $\chi(m) \neq 0$ for all the real $\lambda$.

Note also that equation (1.6) can be established as a condition for existence of a 2π periodic solution by going through the auxiliary equation.

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4 x = 0$$

$\sin ce\ \chi(m) = \left(m^2 - \tfrac{1}{2}a_2\right)^2 + a_4 - \tfrac{1}{4}a_2{}^2$

It is convenient in examining the implication of (1.6) to distinguish the following cases.

(i) $a_2 \leq 0$

(ii) $a_2 = m^2$ for some integer.

From ($i$), it is clear that $a_2 \leq 0$ by equation (1.6)

$$\left.\begin{array}{l} \chi(m) > 0 \\ \text{if} \\ a_4 > 0 \end{array}\right\} \tag{1.7}$$

Here inf $\chi(m) - \chi(N) = a_4 - \tfrac{1}{4}(2N^2)^2 \Rightarrow\ a_4 = N^4$ and so we expect that $\chi(m) > 0$ provided $a_4 > N^4 = \tfrac{1}{4}a_2^2$. The consideration of cases ($i$) and ($ii$) if P is sufficiently small, then a 2π periodic solution of the equation (1.5) exists for arbitrary $a_1$ and $a_3$ if $a_2 \leq 0$ and $a_4 > 0$ or $a_2 = 2N^2$ for some integers

$$N > 0 \text{ and } a_2^2 = 2N^4 \tag{1.8}$$

We now transfer the above consideration to more the general equation

$$x^{(4)} + \varphi(\ddot{x})\dddot{x} + a_2\ddot{x} + \theta(\dot{x}) + f(x) = P(t,x,\dot{x},\ddot{x},\dddot{x}) \tag{1.9}$$

where $\varphi,\ \theta,\ f,\ P$ are continuous functions depending on the arguments shown and $a_2$ is a constant. The equation (1.9) is comparable with (1.5) if

$$\left.\begin{array}{l} \varphi(\ddot{x}) \text{ replaces } a_1 \\ \theta(\dot{x}) \text{ replaces } a_2\dot{x} \\ f(x) \text{ replaces } a_4 x \end{array}\right\} \tag{1.10}$$

The function *f(x)* replacing $a_4 x$ suggest that $f'(x)$ or $x^{-1}f(x)\ (x \neq 0)$ replace $a_4$. So in turn (1.8) suggests that the existence of 2π periodic solutions for (1.9) might be provable for arbitrary $\varphi$, $\theta$ under the following hypotheses. $H_1$: $a_2 \leq 0$ and $f(x)$ subject to the consideration $x^{-1}f(x) > 0$ or if $f'(x)$ exists and $f'(x) > 0$, $H_2$: $a_2 = N^2$ for some integer $N > 0$ with $f(x)$ subject to $x^{-1}f(x) > N^4$ $(x \neq 0)$ or if $f'(x)$ exists and $f'(x) > N^4$.
Thus we have

***Theorem* 1.1**

Suppose in addition to the assumptions on equation (1.9) that given

$$\delta_0 = \begin{cases} 0 \text{ if } a_2 \leq 0 \\ \frac{1}{4}a_2^2, \text{ if } a_2 > 0 \end{cases} \tag{1.11}$$

(i) there exist $\delta_1 > \delta_0$ such that

$$\inf_{\|x\|>1} f'(x) \geq \delta_1 \tag{1.12}$$

and

$$\delta_1 > \tfrac{1}{4}a_2^2 \tag{1.13}$$

(ii) the function P is bounded and 2π periodic in *t*, then equations (1.1) - (1.2) have at least one 2π periodic solution for arbitrary $\varphi$ and $\theta$.

## 2.0 General comments on some notations

Throughout the proof which follows, the capitals $C$, $C_1$, $C_2$, $C_3$,… represent positive constants whose magnitude depend at most on φ, *f*, θ, *P* and $a_2$. The $C_1$, $C_2$, $C_3$,… with suffixes attached retain their identities throughout the proof of theorem 1.1, but the *C's* without suffixes are not necessarily the same in each place of occurence. The symbols $|.|_\infty$, $|.|_1$, and $|.|_2$ in respect of the mappings [0:2π→R] shall have their usual meaning. Thus given the function θ:[0,2π→R] then

$$|\theta|_\infty = \max_{0\leq t\leq 2\pi}|\theta(t)|,\ |\theta|_1 = \int_0^{2\pi}|\theta(t)|dt,\ |\theta|_2 = \left(\int_0^{2\pi}\theta^2(t)dt\right)^{1/2}$$

## 3.0 Proof of Theorem 1.1

The proof of theorem 1.1 is by the Leray-Schauder fixed point technique. See Leray and Schauder [6] and we shall consider the parameter $\lambda$ dependent equation, $(0 \leq \lambda \leq 1)$

$$x^{(4)} + \lambda(\dddot{x})\dddot{x} + a_2\ddot{x} + \lambda\theta(\dot{x}) + f(x) = \lambda P(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{3.1}$$

where, $f_\lambda(x) = (1-\lambda)\delta_1 x + \lambda f(x)$. By setting

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u,\ \dot{u} = -\lambda\varphi u - a_2 z - \lambda\theta(y) - f(x) + \lambda P \tag{3.2}$$

the equation (3.1) can be written compactly in matrix form

$$\dot{X} = AX + \lambda F(X,\ t) \tag{3.4}$$

where

$$X = \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}, A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\delta_1 & 0 & -a_2 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ Q \end{pmatrix} \tag{3.5}$$

with $Q = P(t) - \varphi u - \theta(y) - f(x) + \delta_1 x$

Note that equation (3.1) reduces to a linear equation

$$x^{(4)} + a_2 r^2 + \delta_1 x = 0 \qquad (3.6)$$

when $\lambda = 0$ and to equation (1.9) when $\lambda = 1$. The eigenvalues of the matrix $A$ defined by (3.5) are the roots of the auxiliary equation.

$$r^4 + a_2 r^2 + \delta_1 = 0 \qquad (3.7)$$

The equation (3.7) has no root of the form $r = i\beta$ ($\beta$ an integer), if

$$\delta_1 > \tfrac{1}{4} a_2^2 \qquad (3.8)$$

Therefore the matrix $\left(\ell^{-2\pi A} - I\right)$, ($I$ being the identity 4 × 4 matrix) is invertible. Thus, $X = X(t)$ is a $2\pi$ periodic solution of (3.4) if and only if (Hale, [5])

$$X = \lambda TX,\ 0 \le \lambda \le 1 \qquad (3.9)$$

where the transformation $T$ is defined by

$$(TX)(t) = \int_0^{2\pi} \left(\ell^{-2\pi A} - I\right)^{-1} \ell^{A(t-s)} F\left(X(t), S\right) dt \qquad (3.10)$$

Let $S$ be the space of all continuous 4-vector function $\bar{X}(t) = \left(x(t),\ y(t),\ z(t),\ u(t)\right)$ which are of period $2\pi$ and with norm

$$\left\|\bar{X}\right\|_s = \sup_{0 \le t \le 2\pi} \left\{|x(t)| + |y(t)| + |z(t)| + |u(t)|\right\} \qquad (3.11)$$

If the operator $T$ defined by (3.10) is a compact mapping of $S$ into itself then it suffices for the proof of theorem 1.1 to establish a priori bounds $C_{13}$, $C_9$, $C_8$, $C_{17}$, independent of $\lambda$ such that

$$|x|_\infty \le C_{13},\ |\dot{x}|_\infty \le C_9,\ |\ddot{x}|_\infty \le C_8,\ |\dddot{x}|_\infty \le C_{17} \qquad (3.12)$$

See Scheafer [7].

## 4.0 Verification of (3.12)

Let $x(t)$ be a possible $2\pi$ periodic solution of equation (3.1). The main tool to be used here in this verification is the function $V(x, y, z, u)$ defined by

$$V = \lambda \int_0^s s\varphi(s) ds + a_2 yz + uz + y f_\lambda(x) + \lambda \int_0^y \theta(s) ds \qquad (4.1)$$

The time derivative of $\dot{V}$ along the solution path of (3.2) is

$$\dot{V} = u^2 + a_2 yu + y^2 f_\lambda'(x) + \lambda zP \qquad (4.2)$$

$$\equiv \left(u + \tfrac{1}{2} a_2 y\right)^2 + y^2 \left(f_\lambda'(x) - \tfrac{1}{4} a_2^2\right) + \lambda zP \qquad (4.3)$$

In dealing with the term $y^2 f_\lambda'(x)$, in which $f_\lambda'(x)$ is positive only when $|x|$ is large, consider the function $W$ defined by

$$W = yH(x) \qquad (4.4)$$

where

$$H(x) = \begin{cases} \sin\left(\frac{\pi}{4}\right), & |x| \le 2 \\ \sin x, & |x| > 2 \end{cases} \qquad (4.5)$$

and along the solution paths of (3.2)

$$\frac{d}{dt}\left(yH(x)\right) = y^2 H'(x) + zH(x) \qquad (4.6)$$

By considering the function

$$U = V + \lambda C_0 H(x) \qquad (4.7)$$

along the solution paths (1.7)

$$\frac{d}{dt}(U) \equiv \dot{U} = \left(u + \tfrac{1}{2}a_2 y\right)^2 + y^2\left(f_\lambda'(x) - \tfrac{1}{4}a_2^2\right) + \lambda zP + \lambda C_0 y^2 H'(x) + \lambda C_0 zH(x) \qquad (4.8)$$

Since $|H| \le 2\ \forall x$ and $H'(x) \ge \dfrac{\pi}{4\sqrt{2}}$ when $|x| \le 1$ it follows from (4.8) and $C_0$ is fixed and large enough. Then for every possible 2π periodic solution of (1.9) that

$$\int_0^{2\pi} \left(u + \tfrac{1}{2}a_2 \dot{y}\right)^2 dt + C_1 \int_0^{2\pi} y^2 dt \le C_2 \int_0^{2\pi} |z| dt \qquad (4.9)$$

we have used the boundedness of *P* here. The result (4.8) is the key to the proof of (3.12) but we also need the following inequality

$$|\ddot{x}|_2 \le C_3 |\dddot{x} + \alpha \dot{x}| \qquad (4.10)$$

from Ezeilo and Omari [4].

Now recall that $y = \dot{x},\ z = \ddot{x},\ u = \dddot{x}$. Then

$$|\dddot{x} + \alpha \dot{x}|_2^2 \le C_2 |\ddot{x}|_1 \qquad (4.11)$$

(from 4.9) where $\alpha = \tfrac{1}{2}a_2$, by definition

$$|\ddot{x}|_1 = \int_0^{2\pi} |\ddot{x}| dt \ \le (2\pi)^{1/2} \left(\int_0^{2\pi} \ddot{x}^2 dt\right)^{1/2} = (2\pi)^{1/2} |\ddot{x}|_2$$

By Schwartz's inequality, $\le (2\pi)^{1/2} C_3 |\dddot{x} + \alpha \dot{x}|_1$ by (4.9). Thus, $|\ddot{x}|_1 \le C_4 |\dddot{x} + \alpha \dot{x}|_1$ or $(2\pi)|\ddot{x}|_2 \le C_4 |\dddot{x} + \alpha \dot{x}|_1 = C_5 |\dddot{x} + \alpha \dot{x}|_1$

$$|\ddot{x}_2| \le C_6 |\dddot{x} + \alpha \dot{x}|_2 \qquad (4.12)$$

by Schwartz's inequality. From (4.11), $|\dddot{x} + \alpha \dot{x}|_2^2 \le C_2 C_6 |\dddot{x} + \alpha \dot{x}|_2$ which implies that

$$|\dddot{x} + \alpha \dot{x}|_2 \le C_7 \qquad (4.13)$$

By (4.12) and (4.13), $|\ddot{x}|_2 \le C_8$ and $|\ddot{x}|_\infty \le C_8$ (4.14)

Since *x*(0) = *x*(2π), implies that there τ □ [0,2π] such that exits □(τ) = 0. Then the identity

$$\dot{x}(t) = \dot{x}(\tau) + \int_0^{2\pi} \ddot{x}(s) ds$$

holds. Thus,

$$\max_{0 \le t \le 2\pi} |\dot{x}(t)| \le \int_0^{2\pi} |\ddot{x}| ds$$

$$\le (2\pi)^{1/2} |\ddot{x}|_2$$

by Schwartz's inequality. From (4.14), $\max_{0 \le t \le 2\pi} |\dot{x}(t)| \le (2\pi)^{1/2} C_8$, $\max_{0 \le t \le 2\pi} |\dot{x}(t)| \le C_9$. Thus

$$|\dot{x}|_\infty \le C_9 \qquad (4.15)$$

Now integrate equation (3.1) directly from *t = 0* to *t = 2π*

$$\int_0^{2\pi} x^{(4)} dt + \int_0^{2\pi} \lambda \varphi(\ddot{x}) \dddot{x} dt + \int_0^{2\pi} a_2 \ddot{x} dt + \int_0^{2\pi} \lambda \theta(\dot{x}) dt + \int_0^{2\pi} f_\lambda(x) dt = \int_0^{2\pi} \lambda P dt$$

Using the equation (4.14) and (4.15) and (1.2)

$$\int_0^{2\pi} f_\lambda(x) dt = \int_0^{2\pi} \lambda P dt - \int_0^{2\pi} \lambda \theta(\dot{x}) dt \qquad (4.16)$$

The boundedness of *P* and the fact that $0 \le \lambda \le 1$ together with (4.15) imply that the right hand side of (4.16) is finite. That is, $\left|\int_0^{2\pi} \lambda P dt\right| + \left|\int_0^{2\pi} \lambda\theta(\dot{x})dt\right| \le C_{10}$ (4.17)

Thus, $\left|\int_0^{2\pi} f_\lambda(x)dt\right| \le C_{10}$. So, $\left|(1-\lambda)\,\delta_1 x + \lambda f(x)\right| \le C_{10}$ (4.18)

For $C_{10}$ very large implies that, $|x(t)| \le C_{11}$, for some $\tau \in [0,\ 2\pi]$ (4.19)

Now the identity $t$, $x(t) = x(\tau) + \int_\tau^t \dot{x} ds$ holds. Thus,

$$\max_{0\le t\le 2\pi} |x(t)| \le |x(\tau)| + \int_0^{2\pi} |\dot{x}| ds$$

$$\le C_{11} + (2\pi)^{1/2} \left(\int_0^{2\pi} \dot{x}^2 dt\right)^{1/2}$$

by Schwartz's inequality. From (4.15), $\max_{0\le t\le 2\pi} |x(t)| \le C_{11} + C_{12}$ (4.20)

and $|x|_\infty \le C_{13}$. To obtain the fourth inequality in equation (3.12), multiply equation (3.1) by $x^{(4)}$ and integrate with respect to $t$ from $t = 0$ to $t = 2\pi$

$$\int_0^{2\pi} {x^{(4)}}^2 dt + \int_0^{2\pi} \lambda\varphi(\ddot{x})\dddot{x}x^{(4)}dt + \int_0^{2\pi} a_2\ddot{x}x^{(4)}dt + \int_0^{2\pi} \lambda\theta(\dot{x})x^{(4)}dt + \int_0^{2\pi} f_\lambda(x)x^{(4)}dt = \int_0^{2\pi} \lambda P x^{(4)}dt$$

We use equations (4.15), (4.14), (4.20) and the boundedness of $P$ and since $\varphi$, $\theta$ and $f$ are continuous functions, there are constants $C_{14}$, $C_{15}$ such that

$$\left|x^{(4)}\right|_2^2 \le C_4 \left|x^{(3)}\right|_2 \left|x^{(4)}\right|_2 + C_{15}\left|x^{(4)}\right|_2 \le C_{16}\left|x^{(4)}\right|_2 \quad (4.21)$$

where $C_{16} = C_{15} + C_{15}C_7$ (from 4.13). Hence, $\left|x^{(4)}\right|_2 \le C_{15}$. From which because of (1.2) with $r = 3$, then

$$\left|x^3\right|_\infty \le (2\pi)^{1/2} C_{16} = C_{17} \quad (4.22)$$

which is the fourth inequality in (3.12).

## 5.0 Conclusion

The estimates (4.14), (4.15), (4.20) and (4.22) verify the inequality (3.12) and hence the proof of theorem 1.1, which implies the existence of at least one $2\pi$ periodic solution for equation (1.1) – (1.2).


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