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Coefficients and constants in the partial fractions of some trigonometric inverse functions

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Abstract

This Paper extend the work done by Daniel and Tella,* [1] *and Ogbereyivwe, Emumejaye and Ekeh,* [3] *to trigonometric inverse functions. It determines the unknown constants and coefficients in resolving rational functions containing trigonometric inverse functions in their denominator, into the sum of its partial fractions equivalent by recursive method. For each of the cases of functions considered, a recursion formular was derived and the trend of these constants and coefficients were examined as n tends to infinity.

1.0 Introduction

Resolving rational functions with inverse function of trigonometric functions into sum of simpler rational functions is tasking and stimulating particularly in the area of Calculus and Applied Mathematics. Researchers may encounter problems of this nature or problems that can be transformed into the form we are considering.

Several studies have been carried out on the coefficients in the partial fraction of rational functions with trigonometric functions in its denominator. Daniel and Tella, [1] studied the coefficients in the partial fraction of rational function with $cosx$ in its denominator, while Ogbereyivwe, Emumejaye and Ekeh, [3] studied the case where the denominator contains $sinx$ and exponential function (e^x). These cited studies, neglected the case where the denominator contain trigonometric inverse functions.

The following deductions were made in the earlier work done by Daniel and Tella in [1].

Let
$$R_n x = \frac{1}{(1-x^2)Cosx} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{\sum_{j=0}^{n} D_j x^j}{\sum_j^n (-1)^n \frac{x^{2n}}{(2n)!}} \qquad (1.1)$$

Then, the constants and coefficients in the sum of its Partial fractions equivalent is:

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$$\begin{cases} D_0 = 1 - 2k, \quad A = B = k \\ D_{2j} = \frac{(-1)^{j+1}2k}{(2j)!} + D_{2j-2,} \; j = 1,2, \dots\dots. \\ D_{2j+1} = 0\,, \;\; \forall \; j = 0, 1, 2, \dots\dots.. \\ k = 1 + \frac{1}{2} + \frac{5}{24} + \cdots \end{cases} \tag{1.2}$$

And as n tends to infinity we get

$$\begin{cases} A = B = \frac{1}{2}sec1, D_0 = 1 - sec1 \\ D_{2j} = \pm\frac{(-1)^{j+1}sec1}{(2j)!} + D_{2j-2,} \; j = 1,2, \dots\dots. \\ D_{2j+1} = 0\,, \;\; \forall \; j = 0, 1, 2, \dots\dots.. \end{cases} \tag{1.3}$$

Where the expansion of $Cosx$ is valid for the set of all real values of x. Also, Ogbereyivwe, Emumejaye, and Ekeh in [3] deduced that if

$$R_n x = \frac{1}{(1-x^2)Sinx} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{\sum_{j=0}^{2n-2} D_j x^j}{\sum_0^n \frac{(-1)^n x^{2n}}{(2n+1)!}} \tag{1.4}$$

where the series expansion of $Sinx$ is valid for all real values of x, then the constants and coefficients are :

$$\begin{cases} D_1 = -2k + 1 + \frac{1}{3!}, \quad A = 1, \quad C = -B = k \\ D_{2j-2} = 0, \quad \forall \quad j = 1,2,3,\cdots\cdots\cdots \\ D_{2j-1} = (-1)^{j-1}\frac{2k}{(2j+1)!} + \sum_j^{j+1}(-1)^{j+1}\frac{1}{(2j+1)!} + D_{2j-2}, \\ \quad \forall \quad j = 1,2,3,\cdots\cdots \quad , \quad and \quad k = \frac{1}{2(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots\cdots)} \end{cases} \tag{1.5}$$

and for $\lim_{n\to\infty} R_n(x)$ we get :

$$\begin{cases} D_1 = -2k + 1 + \frac{1}{3!}, A = 1, C = -B = \frac{1}{2}\cos ec\,1 \\ D_{2j-2} = 0, \forall j = 1,2,3,\cdots 1.6 \\ D_{2j-1} = (-1)^{j-1}\frac{2\sin 1}{(2j+1)!} + \sum_j^{j+1}(-1)^{j+1}\frac{1}{(2j+1)!} + D_{2j-2}, \forall j = 1,2,3,\cdots \end{cases} \tag{1.6}$$

and the series expansion of *sinx* is valid for all real values of *x*.

This paper extends the work to cases where the numerator is a constant unit function and the denominator of the rational function, contains trigonometric function like $Sin^{-1}x$, $Cos^{-1}x$, $Cot^{-1}x$ *and* $tan^{-1}x$.

2.0 *The Case* $R_n(x) = \frac{1}{(1-x^2)Sin^{-1}x}$

Consider $$R_n(x) = \frac{1}{(1-x^2)Sin^{-1}x} \tag{2.1}$$

$$= \frac{1}{(1-x^2)\left(x+\sum_{j=0}^{n}\frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=0}^{n}(2j+1)}x^{2j+3}\right)} \quad (see\ [4]) \tag{2.2}$$

The expansion is valid for $|x| < 1$ and $-\frac{\pi}{2} < Sin^{-1}x < \frac{\pi}{2}$, and let $\alpha = \frac{1}{2\cdot 3}$, $\beta = \frac{1\cdot 3}{2\cdot 4\cdot 5}$, $\delta = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 5\cdot 6\cdot 7}$ and so on.

Then, $R_n(x) = \frac{1}{(1-x^2)(x+\alpha x^3+\beta x^5+\delta x^7+\cdots)}$

$$= \frac{1}{x(1-x^2)(1+\alpha x^2+\beta x^4+\delta x^6+\cdots)} \tag{2.3}$$

Resolving the rational function into partial fraction for values of n, we get

When $n = 1$, $R(x) = \frac{1}{x(1-x^2)} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x}$

$$\Longrightarrow \quad A = 1\ , \quad B = -\frac{1}{2}\ , \quad C = \frac{1}{2}$$

When $n = 2$, $R(x) = \frac{1}{x(1-x^2)(1+\alpha x^2)} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D_0+D_1x}{(1+\alpha x^2)}$

Solving for $A, B, C, D_0, and\ D_1$ in above, we have :

$$A = 1\ , \quad B = -\frac{1}{2(1+\alpha)}\ , \quad C = \frac{1}{2(1+\alpha)}$$

$$D_0 = -(B-C) \quad , \quad D_1 = (B-C)-(\alpha-1)$$

When $n = 3$, $R(x) = \frac{1}{x(1-x^2)(1+\alpha x^2+\beta x^4)} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D_0+D_1x+D_1x^2+D_1x^3}{(1+\alpha x^2+\beta x^4)}$

and the constants are as follows :

$$A = 1\ , \quad B = -\frac{1}{2(1+\alpha+\beta)}\ , \quad C = \frac{1}{2(1+\alpha+\beta)}$$

$$D_0 = -(B-C) \quad , \quad D_1 = (B-C)-(\alpha-1)$$

$$D_2 = D_0 - \alpha(B+C), \quad D_3 = D_1 - \alpha(B-C) - (B-\alpha)$$

When $n = 4$, $R(x) = \frac{1}{x(1-x^2)(1+\alpha x^2+\beta x^4+\delta x^6)}$

$$\equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D_0+D_1x+D_1x^2+D_1x^3}{(1+\alpha x^2+\beta x^4)}$$

and the constants are as follows :

$$A = 1\ , \quad B = -\frac{1}{2(1+\alpha+\beta+\delta)}\ , \quad C = \frac{1}{2(1+\alpha+\beta+\delta)}$$

$$D_0 = -(B-C) \quad , \quad D_1 = (B-C)-(\alpha-1)$$

$$D_2 = D_0 - \alpha(B+C), \quad D_3 = D_1 - \alpha(B-C) - (B-\alpha)$$

$$D_4 = D_2 - \beta(B-C), \quad D_5 = D_3 - \beta(B-C) - (\delta-B)$$

Continuing in this manner we draw the following conclusion:

$$R_n(x) = \frac{1}{(1-x^2)Sin^{-1}x} = \frac{1}{(1-x^2)\left(x+\sum_{j=0}^{n}\frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)}x^{2j+3}\right)}$$

$$\equiv \frac{B}{1+x} + \frac{C}{1-x} + \frac{\sum_{j=0}^{n} D_j x^j}{x+\sum_{j=0}^{n}\frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=1}^{n-1}(2j+1)}x^{2j+3}} \tag{2.4}$$

where

$$\begin{cases} A = 1 \\ B_n = \dfrac{1}{2\sum_0^n \frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)}} \\ C_n = \dfrac{1}{2\sum \frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)}} \\ D_0 = -(B_n - C_n) \\ D_1 = (B_n - C_n) - (\alpha - 1) \\ D_{2n} = D_{2n-2} - \frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)}, \forall\ n = 2,3,4,\ldots \\ D_{2n-1} = D_{2n-3} - \frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)}(\Psi),\ \forall\ n = 2,3,4,\ldots \\ where\ \ \Psi = \left(\frac{\prod_{j=0}^{n-1}(2j+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)} - \frac{\prod_{j=0}^{n-2}(2j+1)}{(2n+1)\prod_{j=0}^{n-2}(2j+2)}\right) \end{cases} \tag{2.5}$$

By observing the trend of the constants and coefficients as $n \to \infty$, we get,

$$R_\infty(x) = \frac{1}{(1-x^2)Sin^{-1}x} = \frac{1}{(1-x^2)\left(x + \sum_{j=0}^{\infty} \frac{\prod_{j=0}^{\infty}(2j+1)}{(2n+1)\prod_{j=0}^{\infty-1}(2j+2)} x^{2j+3}\right)}$$

$$\equiv \frac{B}{1+x} + \frac{C}{1-x} + \frac{\sum_{j=0}^{\infty} D_j x^j}{x + \sum_{j=0}^{\infty} \frac{\prod_{i=0}^{\infty}(2i+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)} x^{2j+3}} \tag{2.6}$$

where

$$\begin{cases} A = 1 \\ B_\infty = \dfrac{1}{2\sum \frac{\prod_{j=0}^{\infty}(2j+1)}{(2n+1)\prod_{j=0}^{n-1}(2j+2)}} \\ C_\infty = \dfrac{1}{2\sum \frac{\prod_{j=0}^{\infty}(2j+1)}{(2n+1)\prod_{j=0}^{\infty-1}(2j+2)}} \\ D_0 = -(B_\infty - C_\infty) \\ D_1 = (B_\infty - C_\infty) - (\alpha - 1) \\ D_{2\infty} = D_{2\infty-2} - \frac{\prod_{j=0}^{\infty-2}(2j-1)}{(2n+1)\prod_{j=0}^{n-2}(2j+2)}, \qquad \forall\ n = 2,3,4,\ldots \\ D_{2\infty-1} = D_{2\infty-3} - \frac{\prod_{j=0}^{\infty-2}(2j+1)}{(2n+1)\prod_{j=0}^{\infty-2}(2j+2)}(\Psi),\ \forall\ n = 2,3,4,\ldots \\ where\ \ \Psi = \left(\frac{\prod_{j=0}^{\infty-1}(2j+1)}{(2\infty+1)\prod_{j=0}^{\infty-2}(2j+2)} - \frac{\prod_{j=0}^{\infty-2}(2j+1)}{(2n+1)\prod_{j=0}^{\infty-2}(2j+2)}\right) \end{cases} \tag{2.7}$$

3.0 *The case* $R_n(x) = \frac{1}{(1-x^2)cos^{-1}x}$

Let $$R_n(x) = \frac{1}{(1-x^2)cos^{-1}x} \tag{3.1}$$

$$= \frac{1}{(1-x^2)\left(\frac{\pi}{2} - \left(x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \cdots\cdots\right)\right)} \quad (3.2)$$

for (3.2), see [4].

$$= \frac{1}{(1-x^2)\left(\frac{\pi}{2} - x - \sum_{i=0}^{n} \frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=1}^{n-1}(2i+2)} x^{2i+3}\right)} \quad (3.3)$$

The expansion is valid for $|x| < 1$ and $0 < cos^{-1}x < \pi,$ and let $\alpha = \frac{\pi}{2}, \beta = \frac{1}{2\cdot 3}$, $\gamma = \frac{1\cdot 3}{2\cdot 4\cdot 5}$, $\delta = \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 5\cdot 6\cdot 7}$ and so on, then (3.3) become:

$$R_n(x) = \frac{1}{(1-x^2)\left(\alpha - (x + \beta x^3 + \gamma x^5 + \delta x^7 + \cdots)\right)} \quad (3.4)$$

We resolve the rational function (3.4) into partial fractions for values of n as follows:

When $n = 1$

$$R(x) = \frac{1}{\alpha(1-x^2)} \equiv \frac{A}{1+x} + \frac{B}{1-x}$$

$$\Longrightarrow \qquad A = -\frac{1}{2\alpha} \ , \ B = \frac{1}{2\alpha}$$

when $n = 2$

$$R(x) = \frac{1}{(1-x^2)(\alpha - x)} \equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{\alpha - x}$$

$$\Longrightarrow \quad A = -\frac{1}{2(\alpha+1)} \ , \quad B = \frac{1}{2(\alpha-1)} \quad and \quad C = \frac{1}{1-\alpha^2}$$

when $n = 3$

$$R(x) = \frac{1}{(1-x^2)(\alpha - x - \beta x^3)} \equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{D_0 + D_1 x + D_2 x^2}{\alpha - x - \beta x^3}$$

then $A = -\frac{1}{2(1+\alpha+\beta)}$, $B = \frac{1}{2(-1+\alpha-\beta)}$ and $D_0 = 1 - \alpha(A+B)$

$$D_1 = A(1+\alpha) + B(1-\alpha) \qquad D_2 = D_0 - (A-B)$$

When $n = 4$

$$R(x) = \frac{1}{(1-x^2)(\alpha - x - \beta x^3 - \gamma x^5)} \equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{D_0 + D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4}{\alpha - x - \beta x^3 - \gamma x^5}$$

And, $A = -\frac{1}{2(1+\alpha+\beta+\gamma)}$, $B = \frac{1}{2(-1+\alpha-\beta-\gamma)}$ and $D_0 = 1 - \alpha(A+B)$

$$D_1 = A(1+\alpha) + B(1-\alpha)$$

$$D_2 = D_0 - (A-B)$$

$$D_3 = D_1 + \beta(A+B)$$

$$D_4 = D_2 - \beta(A-B)$$

When $n = 5$

$$R(x) = \frac{1}{(1-x^2)(\alpha - x - \beta x^3 - \gamma x^5 - \delta x^7)}$$

$$\equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{D_0 + D_1 x + D_2 x^2 + D_3 x^3 + D_4 x^4 + D_5 x^5 + D_6 x^6}{\alpha - x - \beta x^3 - \gamma x^5 - \delta x^7}$$

and

$$A = -\frac{1}{2(1+\alpha+\beta+\gamma+\delta)},$$

$$B = \frac{1}{2(-1+\alpha-\beta-\gamma-\delta)}$$

$$D_0 = 1 - \alpha(A + B)$$
$$D_1 = A(1 + \alpha) + B(1 - \alpha)$$
$$D_2 = D_0 - (A - B)$$
$$D_3 = D_1 + \beta(A + B)$$
$$D_4 = D_2 - \beta(A - B)$$
$$D_5 = D_3 + \delta(A + B)$$
$$D_6 = D_4 - \delta(A - B)$$

Continuing in this manner, we conclude as follows:

$$R_n(x) = \frac{1}{(1-x^2)cos^{-1}x} = \frac{1}{(1-x^2)\left(\alpha - x - \sum_{i=0}^{n}\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=1}^{n-1}(2i+2)}x^{2i+3}\right)}$$
$$\equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{\sum_{i=0}^{n} D_i x^i}{\alpha - x - \sum_{i=0}^{n}\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=0}^{n-1}(2i+2)}x^{2i+3}} \tag{3.5}$$

where

$$\begin{cases} A_n = -\frac{1}{2\left(\alpha + \sum_{i=0}^{n}\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=0}^{n-1}(2i+2)}x^{2i+3}\right)} \\ A_n = \frac{1}{2\left(\alpha - \sum_{i=0}^{n}\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=0}^{n}(2i+2)}x^{2i+3}\right)} \\ D_0 = 1 - \alpha(A_n + B_n) \\ D_1 = A_n(1 + \alpha) + B_n(1 - \alpha) \\ D_2 = D_0 - (A_n + B_n) \\ D_{2n-1} = D_{2n-3} + \left(\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=0}^{n-1}(2i+2)}\right)(A_n + B_n) \\ D_{2n} = D_{2n-2} - \left(\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=0}^{n}(2i+2)}\right)(A_n - B_n) \end{cases} \tag{3.6}$$

By observing the trend of the constants and coefficients as $n \to \infty$. Hence , we get

$$R_\infty(x) = \frac{1}{(1-x^2)cos^{-1}x} = \frac{1}{(1-x^2)\left(\alpha - x - \sum_{i=0}^{\infty}\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=1}^{n}2i}x^{2i+3}\right)}$$
$$\equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{\sum_{i=0}^{\infty} D_i x^i}{\alpha - x - \sum_{i=0}^{\infty}\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=1}^{n-1}2i}x^{2i+3}} \tag{3.7}$$

where

$$\begin{cases} A_\infty = -\frac{1}{2\left(\alpha + \sum_{i=0}^{\infty}\frac{\prod_{i=0}^{\infty-1}(2i+1)}{(2n+1)\prod_{i=1}^{n}2i}x^{2i+1}\right)} \\ A_n = \frac{1}{2\left(\alpha - \sum_{i=0}^{\infty}\frac{\prod_{i=0}^{n-1}(2i+1)}{(2n+1)\prod_{i=1}^{\infty-1}2i}x^{2i+1}\right)} \\ D_0 = 1 - \alpha(A_\infty + B_\infty) \\ D_1 = A_\infty(1 + \alpha) + B_\infty(1 - \alpha) \\ D_2 = D_0 - (A_\infty + B_\infty) \\ D_{2n-1} = D_{2n-3} + \left(\frac{\prod_{i=0}^{\infty-1}(2i+1)}{(2n+1)\prod_{i=1}^{\infty}2i}\right)(A_\infty + B_\infty) \\ D_{2n} = D_{2n-2} - \left(\frac{\prod_{i=0}^{\infty-1}(2i+1)}{(2n+1)\prod_{i=1}^{\infty-1}2i}\right)(A_\infty - B_\infty) \end{cases} \tag{3.8}$$

4.0 *The case* $\boldsymbol{R_n(x) = \frac{1}{(1-x^2)tan^{-1}x}}$

Suppose $$R(x) = \frac{1}{(1-x^2)tan^{-1}x} \quad (4.1)$$

$$= \frac{1}{(1-x^2)\sum_0^n \frac{(-1)^n x^{2n+1}}{2n+1}} \quad see[4] \quad (4.2)$$

$$= \frac{1}{x(1-x^2)\sum_1^n \frac{(-1)^{i-1}x^{2i-2}}{2n-1}} \quad (4.3)$$

We resolve (4.3) into partial fraction for the following values of n.

when $n = 1$ $\quad A = 1, \quad B = -\frac{1}{2}, \quad C = \frac{1}{2}$

when $n = 2$ $\quad A = 1, \quad B = -k, \quad C = k, \quad where \quad k = \frac{1}{2(1-\frac{1}{3})}$

$$D_0 = 0, \quad D_1 = -2k + 1 + \frac{1}{3}$$

when $n = 3$ $\quad A = 1, \quad B = -k, \quad C = k, \quad where \quad k = \frac{1}{2(1-\frac{1}{3}+\frac{1}{5})}$

$$D_0 = 0, \quad D_1 = -2k + 1 + \frac{1}{3}, \quad D_2 = 0$$

when $n = 4$ $\quad A = 1, \quad B = -k, \quad C = k, \quad where \quad k = \frac{1}{2(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7})}$

$$D_0 = D_2 = D_4 = 0$$

$$D_1 = -2k + 1 + \frac{1}{3}, \qquad D_3 = -\frac{2}{3}k - \frac{1}{5} - \frac{1}{3} + D_1$$

$$D_5 = -\frac{2}{5}k + \frac{1}{7} + \frac{1}{5} + D_3$$

when $n = 5$ $\quad A = 1, \quad B = -k, \quad C = k, \quad where \quad k = \frac{1}{2(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9})}$

$$D_0 = D_2 = D_4 = D_6 = D_8 = 0$$

$$D_1 = -2k + 1 + \frac{1}{3}, \qquad D_3 = -\frac{2}{3}k - \frac{1}{5} - \frac{1}{3} + D_1$$

$$D_5 = -\frac{2}{5}k + \frac{1}{7} + \frac{1}{5} + D_3, \qquad D_7 = -\frac{2}{7}k - \frac{1}{9} - \frac{1}{7} + D_5$$

Continuing in this manner, we get

$$R_n(x) = \frac{1}{(1-x^2)tan^{-1}x} = \frac{1}{x(1-x^2)\sum_1^n \frac{(-1)^{i-1}x^{2i-2}}{2n-1}}$$

$$\equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{\sum_{i=0}^{n-1} D_i x^i}{\sum_1^n \frac{(-1)^{i-1}x^{2i-2}}{2n-1}} \quad (4.4)$$

where

$$\begin{cases} D_1 = -2k_n + 1 + \frac{1}{3}, \quad A = 1, \quad C = -B = k_n \\ D_{2j-2} = 0, \quad \forall\ j = 1, 2, 3, \ldots \ldots \ldots \ldots \ldots \ldots . \\ D_{2j-1} = (-1)^{j-1}\frac{2k_n}{2j+1} + \sum_{j=1}^n (-1)^{j+1}\frac{1}{2j+1} + D_{2j-3} \\ k_n = \frac{1}{2\sum_{j=0}^{n+1}\frac{1}{2j-1}} \end{cases} \quad (4.5)$$

and as $n \to \infty$, the constants and the coefficients become

$$R_\infty(x) = \frac{1}{(1-x^2)tan^{-1}x} = \frac{1}{x(1-x^2)\sum_1^\infty \frac{(-1)^{i-1}x^{2i-2}}{2n-1}}$$

$$\equiv \frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x}+\lim_{n\to\infty}\left(\frac{\sum_{i=0}^{n-1} D_i x^i}{\sum_1^n \frac{(-1)^{i-1}x^{2i-2}}{2n-1}}\right) \quad (4.6)$$

but $k_\infty = \frac{1}{2}(tan^{-1}x)^{-1}$, thus:

$$\begin{cases} D_1 = -(tan^{-1}x)^{-1} + 1 + \frac{1}{3}\ , \ \ A = 1\ , \ \ C = -B = \frac{1}{2}(tan^{-1}x)^{-1} \\ D_{2j-2} = 0 \ \ , \quad \forall\ j = 1, 2, 3, \dots\dots\dots\dots\dots \\ D_{2j-1} = (-1)^{j-1}\frac{(tan^{-1}x)^{-1}}{2j+1} + \sum_{j=1}^{n}(-1)^{j+1}\frac{1}{2j+1} + D_{2j-3} \end{cases} \quad (4.7)$$

5.0 *The case* $R_n(x) = \frac{1}{(1-x^2)cot^{-1}x}$

Suppose
$$R(x) = \frac{1}{(1-x^2)cot^{-1}x} \quad (5.1)$$

$$= \frac{1}{(1-x^2)\left(\frac{\pi}{2}-x+\frac{x^3}{3}-\frac{1}{5}x^5+\frac{1}{7}x^7-\dots\dots\right)} \ \ see[4] \quad (5.2)$$

The expansion is true for $|x| < 1.$

Let $\alpha = \frac{\pi}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{1}{5}$, $\delta = \frac{1}{7}$ and so on, then (5.2) becomes:

$$= \frac{1}{(1-x^2)(\alpha - x + \beta x^3 - \gamma x^5 + \delta x^7 - \dots\dots)} \quad (5.3)$$

We resolve (5.3) into partial fraction for values of n and we get

For $n = 1$ $\quad A = -\frac{1}{2\alpha}$ *and* $B = \frac{1}{2\alpha}$

when $n = 2$ $\quad A_2 = -\frac{1}{2(\alpha+1)}$, $B_2 = \frac{1}{2(\alpha-1)}$, $C = \frac{1}{1-\alpha^2}$

when $n = 3$ $\quad A_3 = -\frac{1}{2(\alpha+1-\beta)}$, $B_3 = \frac{1}{2(\alpha-1+\beta)}$, $D_0 = 1-\alpha(A_3+B_3)$

$D_1 = A_3(1+\alpha) + B_3(1-\alpha),$ $\quad D_2 = D_0 - (A_3 - B_3)$

when $n = 4$ $\quad A_4 = -\frac{1}{2(\alpha+1-\beta+\gamma)}$, $B_4 = \frac{1}{2(\alpha-1+\beta-\gamma)}$, $D_0 = 1-\alpha(A_3+B_3)$

$D_1 = A_4(1+\alpha) + B_4(1-\alpha),$ $\quad D_2 = D_0 - (A_4 - B_4)$

$D_3 = D_1 - \beta(A_4 + B_4)$, $\quad D_4 = D_2 + \beta(A_4 - B_4)$

when $n = 5$

$A_5 = -\frac{1}{2(\alpha+1-\beta+\gamma-\delta)}$, $B_5 = \frac{1}{2(\alpha-1+\beta-\gamma+\delta)}$, $D_0 = 1-\alpha(A_5+B_5)$

$D_1 = A_5(1+\alpha) + B_5(1-\alpha),$ $\quad D_2 = D_0 - (A_5 - B_5)$

$D_3 = D_1 - \beta(A_5 + B_5)$, $\quad D_4 = D_2 + \beta(A_5 - B_5)$

$D_5 = D_3 + \gamma(A_5 + B_5)$, $\quad D_6 = D_4 - \gamma(A_5 - B_5)$

Continuing in this pattern, we conclude as follows:

$$R_n(x) = \frac{1}{(1-x^2)cot^{-1}x}$$

$$= \frac{1}{(1-x^2)\left(\alpha + \sum_{i=0}^{n}(-1)^{i+1}\frac{x^{2i+1}}{2i-1}\right)} \quad (5.4)$$

$$\equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{\sum_{i=0}^{n} D_i x^i}{\alpha + \sum_1^n \frac{(-1)^{i+1}x^{2i+1}}{2i-1}} \quad (5.5)$$

$where;$

$$\begin{cases} A_n = \frac{-1}{2\left(\frac{\pi}{2}+\sum_{i=1}^{n}(-1)^{i+1}\frac{1}{(2i-1)}\right)} \\ B_n = \frac{-1}{2\left(\frac{\pi}{2}+\sum_{i=2}^{n}(-1)^{i-1}\frac{1}{(2i-1)}\right)} \\ D_0 = 1-\frac{\pi}{2}(A_n - B_n) \\ D_1 = A_n\left(1+\frac{\pi}{2}\right)+B_n\left(1-\frac{\pi}{2}\right) \\ D_2 = D_0-(A_n-B_n) \\ D_{2n-1} = D_{2n-3}+(-1)^{i+1}\frac{1}{2n-1}(A_n+B_n) \quad \forall i = 0,1,2,..,n = 2,3,... \\ D_{2n} = D_{2n-2}+(-1)^{i+1}\frac{1}{2n-1}(A_n-B_n) \quad \forall i = 0,1,2,..,n = 12,3,... \end{cases} \tag{5.6}$$

By observing the trend of the coefficients as $n \to \infty$, we get

$$R_\infty(x) = \frac{1}{(1-x^2)cot^{-1}x}$$

$$= \frac{1}{(1-x^2)\left(\alpha+\sum_{i=0}^{\infty}(-1)^{i+1}\frac{x^{2i+1}}{2i-1}\right)} \tag{5.7}$$

$$\equiv \frac{A}{1+x}+\frac{B}{1-x}+\frac{\sum_{i=0}^{\infty}D_i x^i}{\alpha+\sum_1^{\infty}\frac{(-1)^{i+1}x^{2i+1}}{2i-1}} \tag{5.7}$$

where

$$\begin{cases} A_\infty = \frac{-1}{2\left(\frac{\pi}{2}+\sum_{i=1}^{\infty}(-1)^{i+1}\frac{1}{(2i-1)}\right)} \\ B_\infty = \frac{-1}{2\left(\frac{\pi}{2}+\sum_{i=2}^{\infty}(-1)^{i-1}\frac{1}{(2i-1)}\right)} \\ D_0 = 1-\frac{\pi}{2}(A_\infty - B_\infty) \\ D_1 = A_\infty\left(1+\frac{\pi}{2}\right)+B_\infty\left(1-\frac{\pi}{2}\right) \\ D_2 = D_0-(A_\infty-B_\infty) \\ D_{2n-1} = D_{2n-3}+(-1)^{i+1}\frac{1}{2n-1}(A_\infty+B_\infty) \quad \forall i = 0,1,2,..,n = 2,3,... \\ D_{2n} = D_{2n-2}+(-1)^{i+1}\frac{1}{2n-1}(A_\infty-B_\infty) \quad \forall i = 0,1,2,..,n = 12,3,... \end{cases} \tag{5.8}$$

6.0 Conclusion

The constants and coefficients in the partial fraction of rational functions, with fundamental inverses of functions of trigonometric functions in its denominator were determined by recursive method. The method used, though may not be generalized to all rational functions with product of polynomials other than the one we considered and trigonometric inverse function in its denominator, but could be used as models of solutions for any problem of the same kind or any problem that may be transformed into the case we considered. Also, it could be a spring board to solving problems that are related to the case we considered.

References

[1] Daniel S. and Tella Y., (2007). “Coefficients in the partial fractions of the trigonometrical functions”. The Journal of Mathematical Association of Nigeria, (Abacus). pp 387-396.

[2] Earl W. Swokowski , (1977). “Calculus with Analytic Geometry”. Prindley Weber and Schmidt, incor po-rated, Massachusetts. pp 457-459.

[3] Ogbereyivwe O., K. Emumejaye and K. I. Ekeh., (2008) “On coefficients in the Partial fraction of some trigonometric and exponential functions”. Journal of the Nigerian Association of Mathematical Physics, Volume 13, pp 41 – 48, University of Benin.

[4] William H. Bayer, (1986). “Standard Mathematics Table”. CRC Press, Inc. Boca Raton, Florida. pp 297 - 298.