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Premium adjustment: actuarial analysis on epidemiological models

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Abstract

In this paper, we analyse insurance premium adjustment in the context of an epidemiological model where the insurer's future financial liability is greater than the premium from patients. In this situation, it becomes extremely difficult for the insurer since a negative reserve would severely increase its risk of insolvency, or might cause bankruptcy. This situation might also make many policy holders withdraw from the insurance by simply terminating their premium payments. It is proved that the benefit reserve changes from negative to positive and from concave to convex under the condition stated in Proposition 5.3 of this paper. As the premium tends to optimum premium rate, the local maximum in the first arch approaches the local minimum in the second arch and they all converge at a time point t_m. As a result, the reserve benefit shifts upwards as the premium rate increases. It is concluded that a proper premium rate between initial and optimum premium rates exist in order to fulfil certain reserves requirements and an algorithm to determine this value was developed.

Keywords

Premium rate, Premium payment, Reserve requirement.

1.0 Introduction

The idea of setting up an insurance coverage against infectious or communicable disease is akin to that of covering other contingencies like natural death and destruction of property. Since mortality analysis is based on ratios instead of absolute counts, we now introduce $s(t)$, $i(t)$ and $r(t)$ respectively as functions of the whole population, in each of the classes S, I, R of the epidemiological model discussed later in this paper.

Let $s(t)$, $i(t)$ and $r(t)$ be the probabilities of an individual being susceptible, infected or removed from an infected class at the time spot, t. In an effort to build a bridge between epidemiological and actuarial models, we analyse possible financial arrangements against premium adjustment resulting from medical treatments given to insured patients.

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The traditional life table methods overlook epidemiological dynamics and the dependence between insurance payers and beneficiaries. It consequently violates the fair premium principle assumed in industries. To obviate this deficiency and show that mathematical models can be used in analysing the transmission dynamics and measuring the effectiveness of controlling strategies, as well as modelling financial arrangements against premium adjustment from the insurer to insured patients resulting from medical treatment is the research investigation contained in this paper.

According to Anderson and May [1], over the last century, many contributions to the mathematical modelling of epidemiological and communicable disease have been made by a great number of public health physicians, epidemiological mathematicians and statisticians. There brilliant works range from empirical data analysis to differential equation theory. Allen and Burgin [2] claimed that some have achieved success in clinical data analysis and effective predictions. Barnes and Fulford [3] considered mathematical modelling with case studies under empirical experiment. And Brayer [4] studied the deterministic compartment models in epidemiology. For a complete review of a variety of mathematical and statistical models, interested readers are referred to Hethcoat [6], Mollison et al [7]. Omorogbe and Omoregie [8], in their paper on epidemiological model in actuarial mathematics, have opined that from a social point of view an effective protection against disease depends not only on the development of medical technology to identify viruses and to treat infected patients, but also on a well-design health-care system.

However, this paper extends Omorogbe and Omoregie [8] by examining the possibility of premium adjustment from the insurer as well as giving an algorithm for determining an added-value premium. For better understanding of this paper we give a fairly detailed treatment of the epidemiological compartment model and actuarial analysis in the next section.

2.0 Epidemiological compartment model and actuarial analysis

To model an epidemic in epidemiological studies, a whole population is usually separated into compartment with labels such as *S, I* and *R*. These acronyms are used in different patterns according to the transmission dynamics of the studied disease. Generally speaking, class *S* denotes the group of individuals without immunity, or those susceptible to a certain disease. In an environment exposed to disease like the Niger Delta, some individuals come into contact with the virus. Those infected who are able to transmit the disease are considered in class *I*. Individuals, removed from the epidemic due to either death or recovery after medical treatment are counted in class *R*. This is illustrated in the upper part of Figure 2.1 describing the transfer dynamics among the three compartments.

Another merit of this partition, from an actuarial perspective, is that the three compartments play significantly different roles in an insurance model. As demonstrated in the lower part of figure 2.1, the susceptible individuals who face the risk of being infected in an epidemic each contribute a certain amount of premium to the insurance funds in return for future coverage of medical expenses incurred as a result of infection. During the outbreak, the infected are eligible for claiming benefits for expenditures covered in the policy. Following an individual’s death, a death benefit for funeral and burial expenses would be paid to specified beneficiaries. Interest will accrue on the properly managed insurance funds at a certain rate.

Let us denote the qualitative relations of $S(t)$, $I(t)$, $R(t)$ functions by the following system of differential equations known as the *SIR* model.

$$S'(t) = -\beta S(t) I \frac{(t)}{N}, \; t \geq 0 \tag{2.1}$$

$$I'(t) = \beta S(t) I \frac{(t)}{N} - \alpha I(t), \; t \geq 0 \tag{2.2}$$

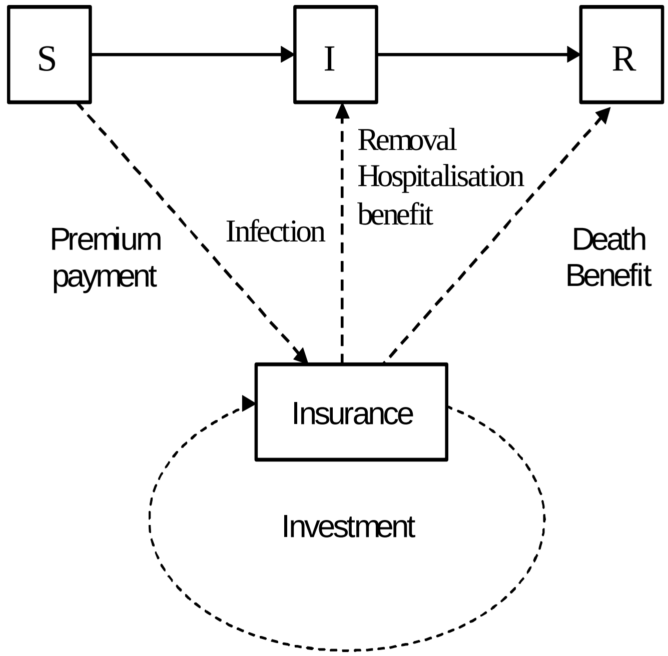

Figure 2.1*:* General transfer dynamics and insurance principle among compartments *S, I* and *R***.**

$$R'(t) = \alpha I(t),\ t \geq 0 \qquad (2.3)$$

with given initial values $S(0) = S_0, I(0) = I_0$ and $S_0 + I_0 = N$

The model is based on the following assumption:

(a) The total number of individuals representing the total population size is kept constant, $N = S(t) + I(t) + R(t)$

(b) Let β be the average number of contact sufficient for an individual to be infected with others per unit time.

(c) At any time a fraction α of the infected leave the class *I* instantaneously. α is also considered to be constant.

(d) There is no entry into or departure from the population, except possibly through death from the disease. For the purpose of setting up an insurance model, the demographic factors like natural births and deaths are negligible, as the time scale of an epidemic is generally shorter than the demographic time scale.

Since the probability of a random contact by an infected person with a susceptible individual is S/N then the instantaneous increase of new infected individuals is $\beta(S/N)I = \beta SI/N$. The third assumption implies that the instantaneous number of people flowing out of the infected class *I* into the removal class *R* is αI.

Let us divide equations (2.1) **-** (2.3) by the constant total population size *N*. Then, we obtain

$$s'(t) = -\beta i(t)s(t),\ t \geq 0, \qquad (2.4)$$

$$i'(t) = \beta i(t)s(t) - \alpha i(t),\ t \geq 0, \qquad (2.5)$$

$$r(t) = 1 - s(t) - i(t),\ t \geq 0, \qquad (2.6)$$

where $s(t)$, $i(t)$ and $r(t)$ are probability functions defined on the interval [0,1]

With these probability functions $s(i)$, $i(t)$ and $r(t)$, we now incorporate actuarial methods to formulate the quantities of interest for an infectious disease insurance.

3.0 Annuity for premium payments and annuity for hospitalization

We assume a premium payment plan in a simple annuity fashion in this work. Individual premiums are collected continuously as long as the covered person remains susceptible, whereas medical expenses are continuously reimbursed to each infected policyholder during the whole period of treatment. Once the individual recovers from the disease, the protection ends right away.

Following the international Actuarial Notation, the actuarial present value (APV) of premium payments from an insured person for the whole epidemic is denoted by $\overline{a}_0^s$ with the superscript indicating payments from class s, and APV of benefit payments from the insurer is denoted by $\overline{a}_0^s$ with the superscript indicating payments to class I.

On the debit side of the insurance product, the total discounted future claim is given by

$$\overline{a}_0^i = \int_0^\infty e^{-\delta t} s(t)dt, \tag{3.1}$$

while on the revenue side, the total discounted future premium is

$$\overline{a}_0^s = \int_0^\infty e^{-\delta t} s(t)dt, \tag{3.2}$$

where δ is the force of interest. Our study in this paper is based on the fundamental *Equivalence principle* in Actuarial Mathematics for the determination of level premiums, which requires:

E[present value of benefits] = *E*[present value of benefits];

Therefore, the level premium for the unit annuity for hospitalisation plan is

$$\overline{P}\left(\overline{a}_0^i\right) = \frac{\overline{a}_0^i}{\overline{a}_0^s} \tag{3.3}$$

Just like in life insurance, where the force of mortality is defined as the additive inverse of the ratio of the derivative of the survival function to the survival function itself, we define here the force of infection as $\mu_t^s = \frac{s'(t)}{s(t)},\ t \geq 0$, and the force of removal as $\mu_t^i = \frac{i'(t)}{i(t)},\ t \geq 0$,

Specifically from (2.4) and (2.5), we see that $\mu_t^i = -\beta i(t)\ and\ \mu_t^i = -\beta s(t) + \alpha.$ Note that the above definitions imply that

$$s(t) = \exp\left\{-\int_0^t \mu_r^s dr\right\} = \exp\left\{-\beta \int_0^t i(r)dr\right\}\ \ t \geq 0, \tag{3.4}$$

and

$$i(t) = \exp\left\{-\int_0^t \mu_r^i dr\right\} = \exp\left\{-\beta \int_0^t s(r)dr + \alpha t\right\}\ \ t \geq 0, \tag{3.5}$$

Proposition 4.1 in the *SIR* model in (2.4) and (2.5),

$$\left(1 + \frac{\alpha}{\delta}\right)\overline{a}_0^i + \overline{a}_0^s = \frac{1}{\delta}. \tag{3.6}$$

Proof

From (2.4) and (2.5), we obtain that $S'(t) + i'(t) = -\alpha i(t),\ t \geq 0$. Integrating wrt r from $t = 0$ to a fixed t gives $s(t) + i(t) - 1 = -\alpha \int_0^t i(r)dr,\ t \geq 0$. Multiplying both sides by $e^{-\delta t}$ and integrating with respect to t from 0 to ∞ yields $\overline{a}_0^s + \overline{a}_0^i - \frac{1}{\delta} = -\frac{\alpha}{\delta}\overline{a}_0^i$. where the right hand side comes from exchanging the order of integrals,

$$\int_0^\infty \exp(-\delta t)\int_0^t i(r)drdt = \frac{1}{\delta}\int_0^\infty \int_0^t i(r)drd(\exp(-\delta t)) = \frac{1}{\delta}\int_0^\infty \exp(-\delta r)i(r)dr = \frac{1}{\delta}\overline{a}_0^i.$$

Notice that the right hand side represents the perpetual annuity. The intuitive interpretation of the left hand side is that if every one in the insured group is rewarded with a perpetual annuity, the APV of expenses from class S accounts for $\overline{a}_0^s$ and similarly that of expenses from class I adds $\overline{a}_0^i$ to the cost. Recall that at any time a fraction α of the infected subgroup move forward to class R. and each of them would receive a perpetual of value 1/δ as well at the time of transition. Therefore, the APV of expenses from this compartment would be $(\alpha/\delta)\overline{a}_0^i$ .It is reasonable that it should sum up to the value of a unit perpetual annuity regardless of the policyholder's location among compartments.

With this relation in mind, we could easily find the net level premium for the unit annuity for hospitalization plan, as follows:

$$\overline{P}(\overline{a}_0^i) = \frac{\overline{a}_0^i}{\overline{a}_0^s} = \frac{\delta \overline{a}_0^i}{1-(\delta+\alpha)\overline{a}_0^i}. \tag{3.7}$$

4.0 Lump sum for hospitalization

The analogy of the plan is with whole life insurance in actuarial mathematics. When a covered person is diagnosed being infected with the disease and hospitalized, the medical expenses is to be paid immediately in lump sum and insurance protection ends. Then the APV of benefit payments to the infected denoted by $\overline{A}_0^i$ can be obtained as

$$\overline{A}_0^i \triangleq \beta \int_0^\infty e^{-\delta t} s(t) i(t) dt, \tag{4.1}$$

since the probability of being newly infected at time t is $\beta s(t) i(t)$

***Proposition* 4.1**

$$\frac{1}{\delta}\overline{A}_0^i + a_0^s = \frac{1}{\delta}s_0, \tag{4.2}$$

and

$$\frac{1}{\delta}i_0 + \frac{1}{\delta}\overline{A}_0^i = \frac{\alpha}{\delta}\overline{a}_0^{1} + \alpha_0^{-i} \tag{4.3}$$

(see proof in Omorogbe and Omoregie [8])

Therefore, for the annuity for hospitalisation plan, the quantitative relations among these insurance factors could be describe by the following ODE systems:

$$\mathrm{P}'(t) = P_{AH} e^{-\delta t} S(t), \quad t > 0 \tag{4.4}$$

$$\mathrm{B}'(t) = e^{\delta t} I(t), \quad t > 0 \tag{4.5}$$

$$V'(t) = P_{AH} e^{\delta t} S(t) - e^{st} I(t), \quad t > 0 \tag{4.6}$$

where $P_{AH} = \overline{P}(\overline{A}_0^i)$ is determined by the equivalence principle. By applying *Runge-Kutta* method of order four (RK-4 method), we should obtain

$$\overline{a}_{0:t7}^s = P(t)/N/P_{AH}, \; \overline{a}_{0:t7}^i = B(t)/N \quad and \quad {}_t\overline{V}(\overline{a}_0^i) = V(t)/N.$$

(See Omorogbe and Omoregie, [8])

5.0 Premium adjustment

Premium is simply the amount of money an insured patient or person pays to the insurer within a specified time interval. The fact that mortality rises with age leads to the consequence that an insurer's future financial liability is always greater than future revenue from benefit premiums. Therefore the benefit reserve is normally positive in traditional life insurance products. Unlike the "U" shape of mortality curve, a unique feature of epidemics is that the infection rate rapidly increases at the beginning and then drops down after reaching a peak. Figure 5.1 illustrates a typical path of a benefit reserve function obtained from the insurance quantities system (4.4) – (4.6), where the benefit premium is determined by the means employed in (3.7).

Although the equivalence principle is applied from time 0 and 5, it is dangerous for an insurer to have a long standing negative reserve, which indicates that so much more expenses are paid out than premiums collected and that the insurer actually becomes a debtor to all policyholders. A negative reserve could severely increase an insurer's risk of insolvency, and in worst scenario might even cause bankruptcy. Another potential hazard is that the insurance policy virtually becomes a certificate of debts. It is likely that a policyholder might withdraw from the insurance simply by stopping payment of premiums. Therefore a prudent insurer would require additional premium in order to keep reserve above an early warning level, which we choose to be zero in our analysis.

Before giving an algorithm for determining an added-value premium, we would like to study for a moment the trend of a benefit reserve function $V(t)$ and its dependency on functions $S(t)$ and $I(t)$.

***Proposition* 5.1**

For the SIR model in (2.1) - (2.3), $S(t)$ is a monotonically decreasing function, and R(t) is a monotonically increasing function; If $S(0) \le \alpha N / \beta,$ then $I(t)$is a monotonically decreasing function. If $S(0) > \alpha N / \beta, I(t)$ increases up to the time when $S(t) = \alpha N / \beta,$ and then decreases after. (see proof in Omorogbe and Omoregie, [8])

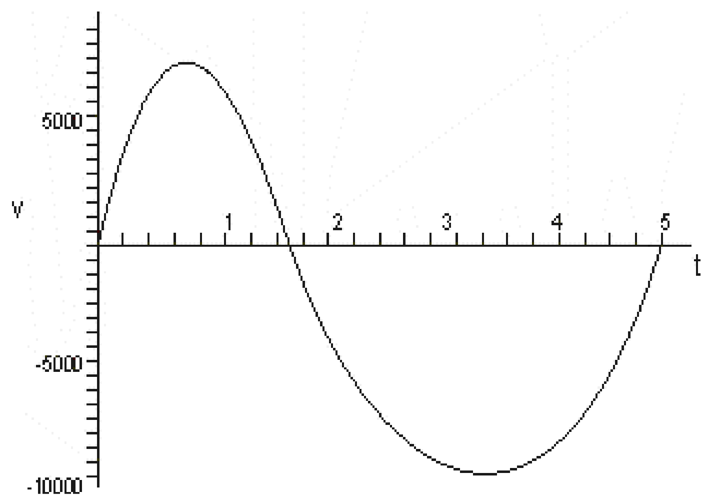

Figure 5.1: Benefit reserve function $V(t)$ for Annuity for Hospitalisation (AH) plan for cancer infection of a community in Niger Delta region of Nigeria (See table in Omorogbe and Omoregie, [8]), $P_{AH} = 106.51$. Double arch structure as explained in Proposition 5.3.

***Proposition* 5.2** (Single Arch Structure)

In the insurance quantities system (4.4) – (4.6), the benefit reverse $V(t)$ is concave, if the premium

$$P_{AH} > \frac{\alpha N}{\beta S\infty} - 1, \tag{5.1}$$

where the constant $c = I_0 + S_0 - \alpha N / \beta \log(S_0)$ *and* $S_\infty = t \xrightarrow{\lim} \infty \; S(t).$

Proof

To check the concavity of $V(t),$ we look at $V''(t),$

$$V''(t) = P_{AH} S'(t) - I'(t) = -\frac{\beta}{N} P_{AH} S(t) \; I(t) - \frac{\beta}{N} S(t) \; I(t) + \alpha I(t)$$

$$= I(t)\left[\alpha - \frac{\beta}{N}(P_{AH} + 1)S(t)\right].$$

It follows that when $P_{AH} > \frac{\alpha N}{\beta S(t)} - 1, \quad \text{for all } t > 0,$ *V(t)* is concave downward. Since *S(t)* is monotonically decreasing, thus condition (5.1) is required.

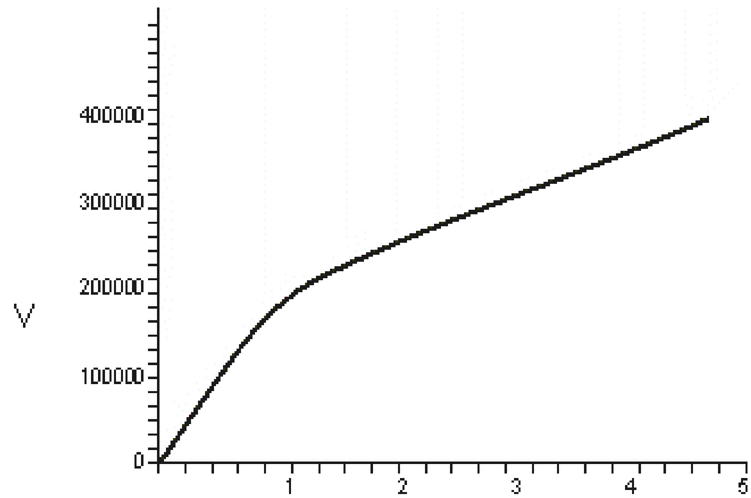

Figure 5.2: Benefit reserve function *V(t)* for AH plan for cancer infection in a community in Niger-Delta region of Nigeria, P_{AH}=843.38, Single arch structure as explained in Proposition 5.2.

***Proposition* 5.3** (Double Arch Structure)

If

$$\frac{\alpha N}{\beta S_0} - 1 < P_{AH} < \frac{\alpha N}{\beta S_\infty} - 1, \qquad (5.2)$$

then the benefit reserve *V(t)* changes from concave to convex, with a point of inflection t_f such that

$$s(t_f) = \frac{\alpha N}{(1 + P_{AH})\beta}. \qquad (5.3)$$

where t_f is an increasing function of the premium rate P_{AH},

and $P_{AH^*} = \frac{I(t_m)}{S(t_m)}$. (see proof in Fend and Garrido, [5])

We would show in the next proposition that t_m is an increasing-then-decreasing function with respect to P_{AH}. Therefore, as the premium rates increase, the local minimum would eventually move backward on the time scale. As one can imagine, when the point of inflection gets closer to the next local minimum, the curve in between becomes flatter. It is a natural conjecture that as the premium P_{AH} rises to a critical value P_{AH}^*, there must be a corresponding time point when t_f overlaps with t_m, as shown in figure 4. thus,

$$\frac{\alpha N}{\beta[1 + I(t_m)/S(t_m)]} = S(t_m),$$

which implies that $S(t_m) + I(t_m) = \alpha N / \beta$. It is not surprising to obtain the same critical value in the following proposition.

***Proposition* 5.4**

For the insurance quantities system in (4.4) – (4.6), the reserve *V(t)* is concave and strictly increasing, if

$$P_{AH} > P_{AH}^{*} = \frac{\alpha N}{\beta}\exp\left(\frac{\beta c}{\alpha N} - 1\right) - 1, \quad (5.4)$$

where the constant $c = I_0 + S_0 - \alpha N/\beta \log(S_0)$.

Proof

To ensure that $V'(t) > 0,$ we need $P_{AH} > \dfrac{I(t)}{S(t)}, \quad \textit{for all } t,$

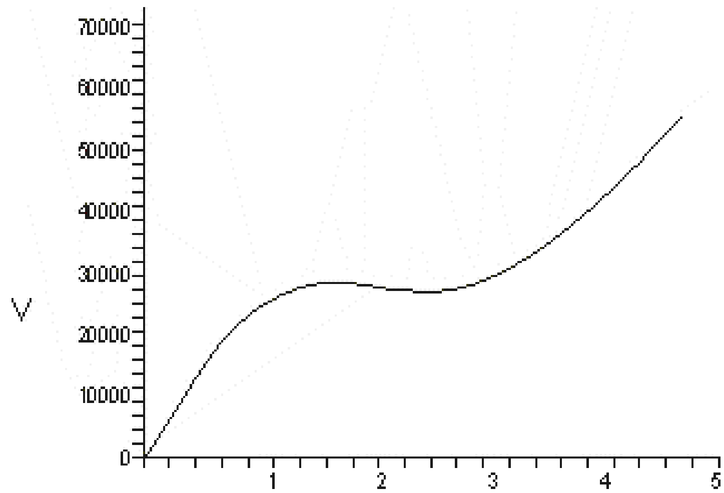

Figure 5.3: Benefit reserve function *V(t)* for *AH* plan for cancer infection in a Niger-Delta community in Nigeria, $P_{AH}=202.17$. Double arch structure and strictly increasing as explained in Proposition 5.4.

or equivalently,

$$\log P_{AH} > \log I(t) - \log S(t), \text{ for all } t.$$

let $$f(t) = \log I(t) - \log S(t),$$

then

$$f'(t) = \frac{I'(t)}{I(t)} - \frac{S'(t)}{S(t)} = \frac{\beta}{N}\left[S(t) + I(t)\right] - \alpha,$$

by (2.1) and (2.2). Since $S(t) + I(t) = N - R(t)$ is monotonically decreasing, at the time t_m when

$$S(t_m) + I(t_m) = \alpha N/\beta, \quad (5.5)$$

$f'(t)$ changes from positive to negative and *f(t)* reaches its maximum at time t_m. Thus P_{AH} is required to be greater than *I(tm)/S(tm)*.

$$\frac{I'(t)}{S'(t)} = \frac{dI(t)}{dS(t)} = \frac{(\beta S(t)/N - \alpha)I(t)}{-\beta S(t)I(t)/N} = -1 + \frac{\alpha N}{\beta S(t)},$$

Integrating to find the orbits of the *(S,I)*-plane gives:

$$I(t) + S(t) - \frac{\alpha N}{\beta}\log S(t) = c, \quad (5.6)$$

where c is a constant of integration for each specific orbit, say, $c = I_0 + S_0 - \alpha N/\beta \log(S_0)$. Combining (5.5) and (5.6), we can solve for *S(t)* and *I(t)*, as

$$S(t_m) = \exp(1 - \frac{\beta c}{\alpha N}, \quad (5.7)$$

$$I(t_m) = \frac{\alpha N}{\beta} - \exp\left(1 - \frac{\beta c}{\alpha N}\right). \quad (5.8)$$

Hence $$\log P_{AH} > f(t_m). \quad (5.9)$$

Substituting (5.7) and (5.8) into (5.9) gives the condition (5.4).

From the above analysis, we realise that as the premium tends to P^*_{AH}, the local minimum in the second arch and the point of inflection move towards each other, which implies that the local maximum in the first arch approaches the local minimum in the second arch as well. They all converge at the time point t_m. therefore $V(t_m)$ should shift upwards as the premium rates increase. We can infer that a proper premium rate between $\overline{P}(\overline{A}^i_{0:t7})$ and P^*_{AH} exists in order to fulfil certain requirements on the reserves. However, it can not be found in a closed algebraic expression. Instead, an easy algorithm can determine the value.

(a) Clearly defined an early warning level which the reserve function should never go below. For example, $V(t) \geq 0,\ for\ all\ t$.

(b) Start by setting premium rate at $P^{(0)} = \overline{P}(\overline{A}^i_{0:t7})$.

(c) Increase the premium each time by a monetary unit, say, $P^{(n)} = P^{(n-1)} + 0.01$.

(d) Calculate the resulting $V(t_m)$, and see if it is greater than zero. If yes, $P^{(n)}$ gives a reasonable adjusted premium. Otherwise, repeat the last step.

(e) By the fair premium principle, a survival benefit should be awarded to the remaining susceptible policy holders when the policy duration t ends. The benefit amount is determined by $V(t)/S(t)$.

6.0 Discussion and conclusion

One of the problems of insurance is a situation where the future financial liability is greater than future revenue from premiums. This is a dangerous condition for any insurer as much more expenses are paid out than premiums collected and the insurer actually becomes a debtor to all policyholders. A negative reserve could severely increase an insurer's risk of insolvency, and in worst scenario might even cause bankruptcy. Another potential hazard is that the insurance policy virtually becomes a certificate of debt. This might make many policyholders withdraw from the insurance simply by stopping payment of premiums. Therefore a prudent insurer would require additional premium in order to keep reserve above an early warning level, which we choose to be zero in our analysis. We were able to prove that t_m is an increasing-then-decreasing function with respect to premium rate P_{AH}. We also observed from (5.3) that t_f is an increasing function of the premium rate i.e. as we increase the premium, the point of inflection gets closer to the next local minimum and the curve in between becomes flatter. It is natural that as the premium P_{AH} rises to a critical value P^*_{AH}, there must be a corresponding time point when t_f overlaps with t_m as shown in Figure 5.3. From the analysis on this paper, it is discovered that the premium tends to P^*_{AH}, the local minimum in the second arch and the point of inflection move towards each other which implies that the local maximum in the first arch approaches the local minimum in the second arch as well. They all converge at the time point t_m. which implies the reserve function $V(t_m)$ should automatically shift as the premium rate increases. It is concluded that a proper premium rate between $\overline{P}(\overline{A}^i_{0:\bar{t}})$ and P^*_{AH} exist in order to fulfill certain reserves requirements. A simplified algorithm to determine this value was developed.

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