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The construction of optimal hedging portfolio strategies of an investor

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Abstract

We consider the process of constructing an optimal hedging portfolio strategies of an investor. This require the hedging out of risks associated with an investor's portfolio process. In order to achieve this, there is the need for portfolio diversification, that is, investing into different number of investment firms. When the returns from a firm falls below expectation, the returns from other firms can be use to complement the loss. We categorised the investor's portfolio into two folds: the initial investment and the capital gain. Our aim is to construct an hedged portfolio process that can capture all the investor's investment in i, i = 1, 2, …, N investment company at time t, using stochastic differential equation for derivative pricing process. We will also describe the dynamic of our stock price using Binomial lattice model. We also intend to apply Hamilton-Jacobi-Bellman,(HJB) equation to derive the optimal values of our trading strategies. We assume in this paper that the investor is risk averse. Therefore, we adopt an exponential utility function known as Constant Absolute Risk Aversion, (CARA) and maximise the expected final utility function of the investor.

Keywords
Portfolio process; Stock pricing; Capital gain; Derivative pricing; Investor wealth, Short position, Hedging.

1.0 Introduction

We consider the optimal investment strategies of an investor who is endowed with initial wealth and want to maximise his or her expected utility of the terminal wealth. Suppose we have stocks and bonds available whose prices are given in (1.4) and (1.6) respectively. Under the trading strategy $\{\varphi(t)\}$ at time t, we have $\varphi(t)$ invested in the stock and the remaining of the asset in bond, then we have the following stochastic differential equation:

$$dH(t) = \varphi(t)\frac{dS(t)}{S(t)} + (H(t) - \varphi(t))\frac{dM(t)}{M(t)}, \tag{1.1}$$

where H(t) is the wealth process of the investor at time t and $\frac{dS(t)}{S(t)}$ and $\frac{dM(t)}{M(t)}$ are define by (1.4) and (1.6) respectively. Since the investor invested his or her short position into N number of investment companies with different rate of returns in both stock and bond markets, we express (1.1) as follows

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$$d\tilde{H}(t)=\sum_{i=1}^{N}[\varphi^i(t)\frac{dS^i(t)}{S^i(t)}+(\tilde{H}(t)-\varphi^i(t))\frac{dM^i(t)}{M^i(t)}], \qquad (1.2)$$

where $\tilde{H}(t)$ is the sum of wealth from all the investment companies at time t.

***Definition* 1.1**

A portfolio process (a trading strategy) $(\varphi(t), (\phi(t))$,

where

$\varphi\ (t)=(\varphi^1(t),\varphi^2(t),...,\varphi^N(t))^T$ and

$\phi\ (t)=(\phi^1(t),\phi^2(t),...,\phi^N(t))^T$

are $\Re^N$ – valued process that are F-predictable, left continuous and which satisfies

$$\int_1^T \|\varphi(s)\|^2\ ds<\infty,\ \int_1^T \|\phi(s)\|^2\ ds<\infty.$$

Note that $\phi(t)=\tilde{H}(t)-\varphi(t)$.

The process $(\varphi'(t),\phi'(t))_{t\geq 0}$ describes an investor's portfolio in i investment company at time t as carried forward through time where $\varphi'(t)$ and $\phi'(t)$ represent the amount of money invested in the stock and bond markets respectively.

Definition 1.2

A comsumption process $\Lambda(t), 0\leq t\leq T$ is a nonnegative and F-adapted process such that $\int_0^T \Lambda(t)dt<\infty.$

This is the total amount consumed by the investment companies from time t = 0 to time t = T on behalf of the investor. The adapted condition means that the investor cannot anticipate the future. Hence, the wealth of the investor at time t is express as follows:

$$dH^{\varphi,\Lambda}(t)=\sum_{i=1}^{N}[\varphi^i(t)\frac{dS^i(t)}{S^i(t)}+(H^{\varphi,\Lambda}(t)-\varphi^i(t))\frac{dM^i(t)}{M^i(t)}]-\int_0^t \Lambda(t)dt. \qquad (1.3)$$

$\varphi^i(t)$ represents the amount invested in the stock market by the investor through investment company $i, 1\leq i\leq N$ at time t and $\phi^i(t)$ represents the amount invested in the bond market by the investor through investment company $i, 1\leq i\leq N$ at time t. $\int_0^t \Lambda(s)ds,$ represents the total amount spent(costs) in the investment period up to time t by the investment companies.

Definition 1.3

Let $r\in\Re$. Then $m(t,T)$ is called the discount factor if $m(t,T) = \exp[-r(T-t)]$ provided $r\geq 0$, and $0 < m(t,T)\leq 1$ almost surely for ever t, $T\in\aleph$ with $t\geq T$. In this case r, represents the deterministic interest rate.

***Definition* 1.4**

The trading strategy $\varphi(t)$ is said to be admissible if the following are satisfies:

(a) $\varphi(t)$ is $\{F_t\}$ adapted, where $\{F_t\}$ is the augmented filtration generated by $W(t)$.

(b) $\varphi(t)$ satisfies the integrability condition $\int_0^T \varphi^2(t)dt<\infty.$

(c) the following Stochastic differential equation admits a unique strong solution:

$$\begin{cases} dH(t)=(rH(t)+(\mu-r)\varphi(t))dt+\varphi(t)\sigma dW(t) \\ H_0=x. \end{cases}$$

***Definition* 1.5**

Let $u_n = 1 + \frac{\sigma}{\sqrt{n}}$ be the upward movements of the stock price after *n* steps per unit time of a Binomial model and $d_n = 1 - \frac{\sigma}{\sqrt{n}}$ be the downward movements after n steps per unit time of a Binomial model. Then, the risk neutral probabilty is given by

$$\tilde{p} = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{\sigma + r/\sqrt{n}}{2\sigma} \text{ and } \tilde{q} = \frac{u_n - 1 - r_n}{u_n - d_n} = \frac{\sigma - r/\sqrt{n}}{2\sigma},$$

where for a single time period of 1/*n*, we set $r_n = r/n$, and nt □ Z represents number of observations.

***Definition* 1.6**

Let X_n be the random variable equal to the number of upward movements of the stock price after *n* steps per unit time of a Binomial model, then the price $S_n(t)$ is

$$S_n(t) = S(0) u_n^{X_n} d_n^{n - X_n},$$

where $n - X_n$ is the number of downward movements of stock price. Substituting in the values of u_n and d_n, into the above equation, we have

$$S_n(t) = S(0) u_n^{X_n} d_n^{n - X_n} = S(0)(1 + \frac{\sigma}{\sqrt{n}})^{\frac{1}{2}(n + X_n)} (1 - \frac{\sigma}{\sqrt{n}})^{\frac{1}{2}(n - X_n)}.$$

This paper is based on previous research work. Black and Scholes [1], constructed a portfolio process that involved one riskless asset and one risky asset. The risk associated with their portfolio process can not be hedged out completely, therefore need for diversification. Merton [6] constructed a discounted portfolio process involving a riskless asset and a risky asset. Hamada [3] constructed a portfolio process that captured one riskless asset and *N* number of risky assets. The investment was spread over different firms. In this case, the risk is at minimal. In their work, the interest rate is considered to be constant and the same for all firms. But, in practise, every investment company has different policies. These policies affect their interest rate. In this paper, we intend to construct and diversify the investor's portfolio into *N* number of stocks and *N* number of bonds with each firms having unique interest rates (different interest rates) and determine the optimal investment strategy of the investor at time *t*. We assume in this paper that the markets are complete and frictionless.

1.2 Our stock pricing process

Our stock pricing process satisfies the stochastic differential equation,

$$\frac{dS^i(t)}{S^i(t)} = \mu_{it} dt + \sum_{j=1}^{K} \sigma_{ijt} dW_j(t), \; i = 1, \ldots, N \tag{1.4}$$

This says that the infinitesimal change $dS^i(t)$ in the stock price at time t under i investment company, as a percentage of the value $S^i(t)$, is given by a drift term $\mu_{it} dt$ and a 'fluctuation' or small movement upwards and downwards given by $\sum_{j=1}^{K} \sigma_{ijt} dW_j(t)$ at time t of i investment company. The equation below solves (1.4):

$$S^i(t) = S^i(0) \exp\left[\mu_i t + \sum_{j=1}^{K} \left(\sigma_{ij} W_j t - \frac{1}{2} \sigma_{ij}^2 t \right) \right], \; i = 1, \ldots, N \tag{1.5}$$

(see Theorem 1.1 below, for the proof). By definition 1.5 and 1.6, we have that

$$\lim_{n\to\infty} S_n^i(t) = S^i(0)\exp\left[\tilde{\mu}_i t + \sum_{j=1}^{K}\left(\sigma_{ij} W_j(t)\ \right)\right], i = 1,\ldots, N$$

where $W(t)$ is a normally distributed random variable with mean zero and variance t. Therefore, the asset pricing process becomes

$$\lim_{n\to\infty} S_n^i(t) = S^i(0)\exp\left[\mu_i t + \sum_{j=1}^{K}\left(\sigma_{ij} W_j t - \frac{1}{2}\sigma_{ij}^2 t\right)\right], i = 1,\ldots, N$$

Since $\lim_{n\to\infty} S_n^i(t) = S^i(t)$, we have that

$$S^i(t) = S^i(0)\exp\left[\mu_i t + \sum_{j=1}^{K}\left(\sigma_{ij} W_j t - \frac{1}{2}\sigma_{ij}^2 t\right)\right], i = 1,\ldots, N$$

This is referred to as Wiener process.

***Theorem* 1.1**

The Wiener model for our asset pricing process with dynamics:

$$S^i(t) = S^i(0)\exp\left[\mu_i t + \sum_{j=1}^{K}\left(\sigma_{ij} W_j t - \frac{1}{2}\sigma_{ij}^2 t\right)\right], i = 1,\ldots, N$$

satisfies the stochastic differential equation

$$\frac{dS^i(t)}{S^i(t)} = \mu_{it} dt \ \ + \sum_{j=1}^{K} \sigma_{ijt} dW_j(t), i = 1,\ldots, N \quad .$$

Proof

In order to establish that (1.5) solve (1.4), we define the stochastic process

$$X(t) = (\mu_i - \frac{1}{2}\sum_{j=1}^{K}\sigma_{i,j}^2)dt + \sum_{j=1}^{K}\sigma_{i,j} dW_j(t), i = 1,\ldots, N$$

Then,

$$dX(t)^2 = \sum_{j=1}^{K}\sigma_{i,j}^2 dt, i = 1,\ldots, N.$$

We now consider the asst pricing process as a function of a process for which we know the differential equation

$$S_t^i(X(t)) = S^i(0)\exp[X(t)]$$

$$d[S_t^i(X(t))] = S^i(0) d(\exp[X(t)])$$

$$= S^i(0)\{\frac{\partial(\exp[X(t)]}{\partial X(t)} dX(t) + \frac{1}{2}\frac{\partial^2(\exp[X(t)])}{\partial X(t)^2} dX(t)^2\}$$

$$= S^i(0)\exp[X(t)](dX(t) + \frac{1}{2} dX(t)^2)$$

$$= S^i(t)(dX(t) + \frac{1}{2} dX(t)^2)$$

$$= S^i(t)(\sum_{j=1}^{K}(\mu_i - \frac{1}{2}\sigma_{i,j}^2)dt + \sum_{j=1}^{K}\sigma_{i,j}dW_j(t) + \frac{1}{2}\sum_{j=1}^{K}\sigma_{i,j}^2 dt = S^i(t)[\mu_i dt \ + \sum_{j=1}^{K}\sigma_{ij}dW_j(t),\ i = 1,\cdots,N$$

as required.
The dynamics of our bond pricing process is given as follows:

$$\frac{dM^i(t)}{M^i(t)} = r_i dt,\ i = 1,\cdots,N \tag{1.6}$$

r_i is the intantaneous interest rate for bond i and $M^i(t)$ is the price process of the bond i at time t for investment company i.

1.3 Dynamics of our derivative pricing

We can now define our derivative pricing dynamics. Let $df(t)$ be our derivative pricing, then for

$$f = \left(f^1(t),\ f^2(t),\cdots,\ f^N(t)\right)^T = \left(f^i(t)\right)^T_{t\geq 0,\ i=1,\cdots N},$$

we have

$$df(t) = \mu_t^f\ f(t)dt\ + \sum_{j=1}^{K}\sigma_{jt}^f\ f(t)dW_j(t)$$

$$\Rightarrow \qquad df^i(t) = \mu_{it}^f\ f^i(t)dt + \sum_{j=1}^{K}\sigma_{ijt}^{f^i}\ f^i(t)dW_j(t),$$

where $\mu_i^{f^i}\ (t)\ and\ \sigma_{ij}^{f^i}\ (t)$ are the drifts and volatilities of our derivative pricing respectively.

***Lemma* 1.1:**
Given that

$$df^i(t) = \mu_i^{f^i}(t)f^i(t)dt\ +\ \sum_{j=1}^{K}\sigma_{ij}^{f^i}(t)f^i(t)dW_j(t).$$

and

$$dS^i(t) = \mu_i(t)S^i(t)dt\ + \sum_{j=1}^{K}\sigma_{ij}(t)S^i(t)dW_j(t), \text{ then for}$$

$$dV(t) = r\ (t)V\ (t)dt\ ,$$

we have

$$\frac{\mu_i^{f^i}(t) - r(t)}{\sigma_{ij}^{f^i}(t)} = \frac{\mu_{i(t)} - r(t)}{\sigma_{ij}(t)}.$$

2.0 Portfolio process of an investor

We are now in a position to show that the investor's portfolio process is self-financing.

***Theorem* 2.1**

The investor's trading strategy (or portfolio process) $(\varphi^i(t),\phi^i(t))_{t\geq 0}$ is self-financing.

Before establishing Theorem (1.2), we need the following results.

***Lemma* 2.1**

If $J(t) = S^i(t)M^i(t), i = 1,...,N,$ then $d(J(t)) = S^i(t)dM^i(t) + M^i(t)dS^i(t) + dS^i(t)dM^i(t).$

Proof

Using $It\hat{o}'s\ lemma,$ we find the product rule $d(S^i(t)M^i(t))$ for stochastic process $S^i(t)\ and\ M^i(t)$. Expanding the differential as a Taylor's Series in $dS^i(t)\ and\ dM^i(t)$ up to second order derivative, we have

$$d(S^i(t)M^i(t)) = \frac{\partial(S^i(t)M^i(t))}{\partial S^i(t)}dS^i(t) + \frac{\partial(S^i(t)M^i(t))}{\partial M^i(t)}dM^i(t) + \frac{1}{2}\frac{\partial^2(S^i(t)M^i(t))}{\partial S^i(t)^2}dS^i(t)^2$$

$$+\frac{1}{2}\frac{\partial^2(S^i(t)M^i(t))}{\partial M^i(t)^2}dM^i(t)^2 + \frac{\partial^2(S^i(t)M^i(t))}{\partial S^i(t)\partial M^i(t)}dS^i(t)dM^i(t)$$

$$= M^i(t)dS^i(t) + S^i(t)dM^i(t) + dS^i(t)dM^i(t),$$

as required.

***Lemma* 2.2**

Let $M^i(t)$ be the bond price and $\phi^i\ (t)$ be the amount invested in the bond market by the investor through the investment company i at time t. Again, let $S^i(t)$ be the stock price and $\varphi^i\ (t)$ be the amount invested in the stock market at time t. Then,

$$\sum_{i=1}^{N} M^i(t)d\phi^i(t) = \sum_{i=1}^{N}(-d\varphi^i(t)S^i(t) - d\varphi^i(t)dS^i(t)).$$

Proof

Suppose that $f^i(t) = \varphi^i(t)S^i(t) + \phi^i(t)M^i(t),$ then $f^i(t) - \varphi^i(t)S^i(t) = \phi^i(t)M^i(t)$.

Hence,

$$\sum_{i=1}^{N} M^i(t)d\phi^i(t) = \sum_{i=1}^{N}(df^i(t) - \varphi^i(t)dS^i(t)) = \sum_{i=1}^{N} d(f^i(t) - \varphi^i(t)S^i(t))$$

$$= M^i(t)[\frac{df^i(t)}{M^i(t)} - \varphi^i(t)\frac{dS^i(t)}{M^i(t)} - \frac{d\varphi^i(t)S^i(t)}{M^i(t)} - \frac{d\phi^i(t)dS^i(t)}{M^i(t)}].$$

By Lemma 1.1, we have

$M^i(t)d\phi^i(t) = -d\varphi^i(t)S^i(t) - d\varphi^i(t)dS^i(t) + d\varphi^i(t)S^i(t)\frac{dM^i(t)}{M^i(t)} + \frac{d\varphi^i(t)dS^i(t)dM^i(t)}{M^i(t)}$. Since $dM^i(t)$ is deterministc, the product of $dM^i(t)$ with any other stochastic differential is always zero. Hence,

$$\sum_{i=1}^{N} M^i(t)d\phi^i(t) = \sum_{i=1}^{N}(-d\varphi^i(t)S^i(t) - d\varphi^i(t)dS^i(t))$$

as required.

We now establish Theorem 2.1 above.

***Proof* of Theorem 2.1**

We are to show that the investor's portfolio process $(\varphi^i(t), \phi^i(t))_{t\geq 0}$ is indeed self-financing. We first take the differential as follows

$df^i(t) = \varphi^i(t)dS^i(t) + d\varphi^i(t)S^i(t) + d\varphi^i(t)dS^i(t) + d\phi^i(t)M^i(t) + \phi^i(t)dM^i(t) + d\phi^i(t)dM^i(t).$ But, $d\phi^i(t)dM^i(t) = 0.$ Applying Lemmas 1.1 and 2.1, we have

$$df(t)=\sum_{i=1}^{N}[\varphi^i(t)dS^i(t)+d\varphi^i(t)S^i(t)+d\varphi^i(t)dS^i(t)+\phi^i(t)dM^i(t)-(d\varphi^i(t)S^i(t)+d\varphi^i(t)dS^i(t).)]$$

$$=\sum_{i=1}^{N}[\varphi^i(t)dS^i(t)+\phi^i(t)dM^i(t)]\text{, as required.} \qquad (2.1)$$

2.1 The capital gain process of the investor

Let $G_{(\varphi,\phi)}$ be the capital gain of the investor. In order to obtain $G_{(\varphi,\phi)}$ of the investor's portfolio, we integrate equation (2.1) as follows:

$$\int_0^t df(s)=\int_0^t \sum_{i=1}^{N}[\varphi^i(s)dS^i(s)+\phi^i(s)dM^i(s)]$$

$$=\sum_{i=0}^{N}\int_0^t\varphi^i(s)dS^i(s)+\sum_{i=1}^{N}\int_0^t\phi^i(s)dM^i(s)=\sum_{i=1}^{N}\int_0^t(\varphi^i(s)dS^i(s)+\phi^i(s)dM^i(s))$$

Therefore, $f(t)=f(0)+\sum_{i=1}^{N}\int_0^t(\varphi^i(s)dS^i(s)+\phi^i(s)dM^i(s))$

$$=f(0)+G_{(\varphi,\phi)}, \qquad (2.2)$$

where $f(0)$ is the value of the portfolio at the initial time, t=0 and $G_{(\varphi,\phi)}$ is the capital gain associated with and $F-$predictable portfolio process $(\varphi^i(t),\phi^i(t))_{t\geq 0}$ defined to be

$$G_{(\varphi,\phi)}=\sum_{i=1}^{N}\int_0^t(\varphi^i(t)dS^i(s)+\phi^i(s)dM^i(s)) \quad =\int_0^t(\varphi(s)^T dS(s)+\phi(s)^T dM(s)).$$

2.2 Wealth process of an investor

Let $H^{\varphi,\Lambda}(t)$ be the wealth process of the investor at time t. Since investor's portfolio process is self financing, we have

$$dH^{\varphi,\Lambda}(t)=\sum_{i=1}^{N}[\varphi^i(t)\frac{dS^i(t)}{S^i(t)}+\phi^i(t)\frac{dM^i(t)}{M^i(t)}]-\Lambda_t dt$$

$$=\sum_{i=1}^{N}[\varphi^i(t)(\mu_i dt+\sum_{j=1}^{K}\sigma_{ij}dW_j(t))+\phi^i(t)r_i dt]-\Lambda(t)dt=\sum_{i=1}^{N}\varphi^i(t)\mu_i dt+\sum_{i=1}^{N}\sum_{j=1}^{K}\varphi^i(t)\sigma_{ij}dW_j(t)+\sum_{i=1}^{N}\phi^i(t)r_i dt-\Lambda_t dt$$

$$=[\sum_{i=1}^{N}(\varphi^i(t)\mu_i+\phi^i(t)r_i)-\Lambda(t)]dt+\sum_{i=1}^{N}\sum_{j=1}^{K}\varphi^i(t)\sigma_{ij}dW_j(t)$$

$$=[\sum_{i=1}^{N}(\varphi^i(t)\mu_i+r_i(H^{\varphi,\Lambda}(t)-\varphi^i(t))-\Lambda(t)]dt+\sum_{j=1}^{K}\sum_{i=1}^{N}\varphi^i(t)\sigma_{ij}dW_j(t)$$

$$=[\sum_{i=1}^{N}(\varphi^i(t)(\mu_i-r_i)+r_iH^{\varphi,\Lambda}(t)-\Lambda(t)]dt+\sum_{j=1}^{K}\sum_{i=1}^{N}\varphi^i(t)\sigma_{ij}dW_j(t)$$

Now, for constant μ *and* σ, we have

$$dH^{\varphi,\Lambda}(t)=\sum[\varphi^i(t)(\mu_i-r_i)+r_iH^{\varphi,\Lambda}(t)-\Lambda(t)]dt+\sum_{i=1}^{N}\sum_{j=1}^{K}\varphi^i(t)\sigma_{i,j}dW_j(t) \qquad (2.3)$$

We define the value function as follows:

$$U(x,t) = \underset{\varphi^i(t)\in X}{Sup}\; E[u(H^{\varphi,\Lambda}(T)) \mid H^{\varphi,\Lambda}(t) = x].$$

where X is the set of admissible policies that are F_s-progressively measurable and satisfying the integrability condidtion

$$E[\sum_{i=1}^{N} \int_0^t \varphi^i(s)^2 ds] < \infty .$$

The Theorems and the Lemma below will enable us to determine the optimal investment strategies of the investor.

***Theorem* 2.2**

Suppose the value function is defined and $V \in C^{2,1}([0,T]) \times R^N$ Then V is a solution of the following second order partial differential equation:

$$\begin{cases} -\dfrac{\partial v}{\partial t} + \underset{u\in U}{Sup}\, G(t,x,u,-\dfrac{\partial v}{\partial x},\dfrac{\partial^2 v}{\partial x^2}) = 0 \\ v|_{t=T} = h(x) \end{cases} \qquad (2.4)$$

where $G(t,x,u,p,P) = -\frac{1}{2}tr(P\sigma(t,x,u)\sigma(t,x,u)^T + \langle p,\mu(t,x,u)\rangle - f(t,x,u)$ for any (t,x,u,p,P) $\in [0,T]\times\Re^N \times U \times \Re^N \times \Re^N$.

This the Hamilton-Jacobi-Bellman equation(HJB). For more details, see Pu [9] page 38-40.

***Theorem* 2.3**

Let

$$V(s,y) = \inf_{u(.)\in u^*[s,T]} J(s,y;u(.)),\; s,y \in [0,T]\times\Re^N$$
$$V(T,y) = h(y),\; y \in \Re^N,$$

then

$$V(s,y) = \inf_{u(.)\in u^*[s,T]} E\{\int_s^T f(t,x(t;s,y,u(.))dt + V(\tau;s,y,u(.))\}$$

Proof

See Yong et al [12] page 180.

***Lemma* 2.3** (Itô's Lemma)

Let be a function for which the partial derivatives $\frac{\partial f(t,x)}{\partial t}, \frac{\partial f(t,x)}{\partial x}$ and $\frac{\partial^2 f(t,x)}{\partial x^2}$ defined and continuous and let $H(t)$ be a Brownian motion. Then, for every $T \geq 0$ we have

$$f(T,H(T)) = F(0,H(0)) + \int_0^T \frac{\partial f(t,H(t))}{\partial t}dt + \int_0^T \frac{\partial f(t,H(t))}{\partial x}dH(t) + \frac{1}{2}\int_0^T \frac{\partial^2 f(t,H(t))}{\partial x^2}dt$$

We can also write this in the following differential form:

$$\partial f(t,H(t)) = \frac{\partial f(t,H(t))}{\partial t}dt + \frac{\partial f(t,H(t))}{\partial x}dH(t) + \frac{1}{2}\frac{\partial^2 f(t,H(t))}{\partial x^2}dt .$$

For the proof, see Oksendal [8] page 44.

By Theorem 2.3, using Itô's Lemma and the classical principle of dynamic programming of (2.3), yields:

$$U(x,t) \geq E[u(H^{\varphi,\Lambda}(t+h), t+h) \mid H^{\varphi,\Lambda}(t) = x]$$

$$= U(x,t) + E[\sum_{i=1}^{N}\int_t^{t+h} \frac{\partial U(H^{\varphi,\Lambda}(s),s)}{\partial t} + \frac{\partial U(H^{\varphi,\Lambda}(s),s)}{\partial x}(r_i H^{\varphi,\Lambda}(s) - \Lambda(s) + (\mu_i - r_i)\varphi^i(t))$$

$$+\frac{1}{2}\sum_{j=1}^{K}\frac{\partial^2 U(H^{\varphi,\Lambda}(s),s)}{\partial x^2}\varphi^i(s)^2\sigma_{i,j}^2]ds \mid H^{\varphi,\Lambda}(t) = x]$$

For convenience, we drop the summation signs i.e $\sum_{i=1}^{N}$ and $\sum_{j=1}^{K}$. Since we are only interested in the final utility, we set $f(t,x,u)$ in Theorem 2.2 to zero. By subtracting $U(x,t)$ from both sides, dividing both sides by h and then allow h to tend to zero, we obtain:

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial U(x,t)}{\partial x}(r_i x - \Lambda(s) + (\mu_i - r_i)\varphi^i(t) + \frac{1}{2}\frac{\partial^2 U(x,t)}{\partial x^2}\varphi^i(t)^2\sigma_{i,j}^2 \leq 0.$$

This yields the HJB equation for the value function:

$$\frac{\partial U(x,t)}{\partial t} + \max_{\varphi^i(t)}\{\frac{\partial U(x,t)}{\partial x}\frac{1}{2}\frac{\partial^2 U(x,t)}{\partial x^2}\varphi^i(t)^2\sigma_{i,j}^2 + \frac{\partial U(x,t)}{\partial x}(\mu_i - r_i)\varphi^i(t)\}$$
$$+ r_i x\frac{\partial U(x,t)}{\partial x} - \Lambda(t)\frac{\partial U(x,t)}{\partial x} = 0. \qquad (2.5)$$

where $\varphi^i(t)$ is admissible strategy.

It can be shown that the value function in (2.5) is smooth and $V(x,t)$ □ $C^{2,1}(\mathbb{R}^N \times [0,T])$; (see Zariphopoulou [14]), then the value function equals the unique smooth solution of the HJB equation. Therefore, the maximum in (2.5) is well-defined and we have that

$$\varphi^i(t) = -\frac{(\mu_i - r_i)}{\sigma_{i,j}^2}\frac{\partial U(x,t)/\partial x}{\partial^2 U(x,t)/\partial x^2}, i = 1,..., N;\ j = 1,\cdots,K \qquad (2.6)$$

Substitute (2.6) into (2.5) and considering the terminal condition, we have the following HJB equation:

$$\begin{cases}\dfrac{\partial U(x,t)}{\partial t} - \dfrac{(\mu_i - r_i)(\partial U(x,t)/\partial x)^2}{2\sigma_{i,j}^2\,\partial^2 U(x,t)/\partial x^2} + r_i x\dfrac{\partial U(x,t)}{\partial x} - \Lambda(t)\dfrac{\partial U(x,t)}{\partial x} = 0 \\ U(x,T) = u(x), i = 1,\cdots,N;\ j = 1,\cdots,K\end{cases} \qquad (2.7)$$

We assumed that investor is risk averse. Therefore, we use CARA utility function. That is, $u(x) = -\exp[-ax]$, $\alpha > 0$, x □ $\mathbb{R}$, where α is the risk averse coefficient. Our aim is to find a solution that satisfy (2.7). We want a solution of the form $V(x,t) = -\exp[-axQ(t)$ $V(x,t) = -\exp[-\alpha x Q(t) + P(t)]$.

By substituting this into (2.7), we obtain the following differential equations that satisfies $Q(t)$ and $P(t)$:

$$\left.\begin{array}{l}\begin{cases}Q'(t) + r_i Q(t) = 0,\ i = 1,\cdots,N \\ Q(T) = 1\end{cases} \\ \begin{cases}P'(t) = \dfrac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2} + \Lambda(t),\ i = 1,\cdots,N;\ j = 1,\cdots,K \\ P(T) = 0\end{cases}\end{array}\right\} \qquad (2.8)$$

Solving (2.8), we obtain

$$Q(t) = \exp[r_i(T-t)] \text{ and } P(t) = -\frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t) - \int_t^T \Lambda(s)ds, \; i = 1,\cdots,N, \; j = 1,\cdots,K.$$

Then, (2.7) becomes

$$V(x,t) = -\exp\{-\alpha x \exp[r_i(T-t)] - \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t) - \int_t^T \Lambda(s)ds\}, \; i = 1,\cdots,N, \; j = 1,\cdots,K \quad (2.9)$$

Hence, the optimal strategy of the investor's wealth process becomes

$$\varphi^i(t) = -\frac{(\mu_i - r_i)}{\sigma_{i,j}^2} \cdot \frac{-\alpha \exp[r_i(T-t). - \exp\{-\alpha x \exp[r_i(T-t)] - ((\mu_i - r_i)^2 / 2\sigma_{i,j}^2)(T-t) - \int_t^T \Lambda(s)ds\}}{\alpha^2 \exp[2r_i(T-t). - \exp\{-\alpha x \exp[r_i(T-t)] - ((\mu_i - r_i)^2 / 2\sigma_{i,j}^2)(T-t) - \int_t^T \Lambda(s)ds\}} =$$

$$\frac{\mu_i - r_i}{\sigma_{i,j}^2} \cdot \frac{\exp[r_i(T-t)]}{\alpha \exp[2r_i(T-t)]} = \frac{\mu_i - r_i}{\sigma_{i,j}^2} \cdot \frac{\exp[-r_i(T-t)]}{\alpha} = \frac{\mu_i - r_i}{\alpha\sigma_{i,j}^2} m(t,T) \quad (2.10)$$

This implies that the money invested by the risk averse investor in the stock market at time *t* is equal to

$$\frac{\mu_i - r_i}{\alpha\sigma_{i,j}^2} m(t,T), \; i = 1,\cdots,N; \; j = 1,\cdots,K.$$

Therefore, the amount of money invested in the bond market at time t is

$$\phi^i(t) = H^{\varphi,\Lambda}(t) - \varphi^i(t) = \frac{\alpha\sigma_{i,j}^2 H^{\varphi,\Lambda}(t) - (\mu_i - r_i)m(t,T)}{\alpha\sigma_{i,j}^2}, i = 1,\cdots,N; \; j = 1,\cdots,K \quad (2.11)$$

***Theorem* 2.4**

If $V(x,t)$ is the solution of (2.7), then

$$\int_0^T \Lambda(s)ds = -P(t) - \sum_{i=1}^N \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t) + \int_0^t \Lambda(s)ds, \; j = 1,\cdots,K.$$

Proof

From (2.8), we have that

$$P'(t) = \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2} + \Lambda(t), \; i = 1,\ldots,N; \; j = 1,\cdots,K$$

Solving this, we obtain

$$P(t) = -\sum_{i=1}^N \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t) - \int_t^T \Lambda(s)ds, \quad j = 1,\cdots,K$$

$$= -\sum_{i=1}^N \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t) - \int_0^T \Lambda(s)ds - \int_t^0 \Lambda(s)ds, \; j = 1,\cdots,K$$

$$= -\sum_{i=1}^N \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t) - \int_0^T \Lambda(s)ds + \int_0^t \Lambda(s)ds, \; j = 1,\cdots,K$$

Hence,

$$\int_0^T \Lambda(s)ds = -P(t) - \sum_{i=1}^N \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t) + \int_0^t \Lambda(s)ds, \quad j = 1,\cdots,K,$$

as required.

***Corollary* 2.1**

Let H(t) be the portfolio value process of the portfolio $(\varphi^i{}_t, \phi^i{}_t)_{t\geq 0}$, then we have that

$$dH(t) = [\sum_{i=1}^{N}(\varphi^i(t)(\mu_i - r_i) + r_i H(t) + \sum_{j=1}^{K}\frac{1}{2}\sum_{i=1}^{N}\varphi^i(t)^2 \sigma_{ij}{}^2]ds$$
.

***Corollary* 2.2**

If corollary 2.1 above holds, then

$$V(x,t) = -\exp\{-\alpha x \exp[r_i(T-t)] - \frac{(\mu_i - r_i)^2}{2\sigma_{i,j}^2}(T-t)\},\ i = 1,\ldots,N;\ j = 1,\ldots,K.$$

***Remark* 2.1**

Corollary 2.1 and 2.2 above hold when the transaction costs of the investment is zero (i.e., when there is no transaction costs).

3.0 Summary and conclusion

We have considered the investment of the investor's short position into different investment firms with different interest rate from the bond market and different returns from stock market. We also derived an optimal investmet strategy that will guide both the investor and the manager of the firms on how to allocate the investor's short position that are available to the markets. By diversifying the short position of the investor, we have that the risks can be hedged out completely from the investor's portfolio. Hence, every investor is advise to diversify his or her investment in order to minimise risks associated with their investment.

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