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A step-by-step approximation and analysis of asymptotic stability properties of solution of retarded system

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Abstract

The special transcendental character of the characteristic equations of the retarded differential systems makes it difficult to analyze the system equation. Researchers have used various acceptable mathematical techniques to address the issue. In this paper, the convergent properties of an integral equation equivalent of a retarded system are used to establish the existence and uniqueness of the solution of retarded differential equations. A step-by-step approximating technique is employed in formulating a numerical method of solving an initial value problem of the retarded system and the solution is presented in the form of a finite series. The asymptotic stability properties of the solution are investigated. Results obtained are comparable to the general solution form of the ordinary differential equations

Keywords

Retarded differential equations, existence and uniqueness, step-by-step approximation, asymptotic stability

1.0 Introduction

In an attempt to analyse real life problems, the concept of mathematical model or formulation of problems are readily employed [4]. The mathematical models often chosen are differential equations. Differential equations merely abstract the reality of dynamic systems by disregarding certain physical facts which seem to be of minor influence, such that in complicated physical situations the differential equation does not guarantee the true picture of reality [10]. The introduction of functional differential equations [1], [2], [3], [4], [6] has helped to address the lapses of the differential equations in modeling dynamic systems.

Retarded equations are special class of functional differential equations with time lag functions incorporated only in the state of the system, which account for the past states as well as the present states [1]. A general retarded functional differential equation is given as,

$$\dot{x}(t) = f\left(t,\ x(t),\ x(t-nh)\right), n = 1,\ 2,\ 3, \cdots, \tag{1.1}$$

where $x(t)$ is the state of the system at time *t*, $\dot{x}(t)$ is the derivative of the state function with respect to time *t*, and $x(t-nh)$ is the time lag function, with h > 0 defining the delay interval.

The challenges of analyzing system (1.1) include the establishment of the theory on the existence and uniqueness of the solution, finding an analytic solution, and analyzing the asymp-

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totic stability properties of the solution [7]. Necessary and sufficient conditions for the existence and uniqueness of the solution of (1.1) are of immense importance, and research to investigate these conditions are found in the works of [4], [5], [6], [7] and [13].

Hale [6] provides necessary and sufficient conditions for global existence and exponential estimates of the solution of non-linear retarded system

$$\begin{aligned} \dot{x}(t) &= L(t,\ x(t-h)) + b(t),\ t > 0 \\ x(t) &= \varphi,\ t \in [t_0 - h, t_0]. \end{aligned} \tag{1.2}$$

This is extended to the linear retarded system (1.1) by Driver [4] and Falbo [5]. The set back in analyzing system (1.1) lies in its special transcendental character of the characteristic equations that makes it difficult to find an analytic solution. [1], [2] and [4] employ the concept of exponential estimate in solving the characteristic equation of (1.1), while [1], [4] and [6] utilize approximating techniques in achieving their results. Different acceptable techniques have been employed to investigate the necessary and sufficient conditions for asymptotic stability properties of the solution of (1.1), see ([1], [3], [4],[6], [8], [9] and [12]).

This paper explores the convergent properties of the integral equation equivalent of (1.1) to establish the existence and uniqueness of solution of the system. Also, a step-by-step approximating technique is used in formulating finite series solution whose asymptotic stability properties are investigated for each delay interval by utilizing local Lipschitzian condition.

1.2 Notations

E^n is an *n*–dimensional Euclidean space for $n > 0$, with $\|\cdot\|$ as the Euclidean vector norm. $B_H([t_0 - h, t], E^n)$ is a Banach space of continuous differentiable function on $[t_0 - h,\ t]$, where { $h : [t_o - h, t] \to E^n$ } and h is continuous . $\varphi(s)$ is a continuous differentiable function with norm in $B_H([t_o - h, t], E^n)$ defined as $\|\varphi(s)\| = \sup_{t_0 - h \le s \le t_0} |\varphi(s)|$, and $x(s) = x(t - h)$, $t \ge t_o$ defines the trajectory segment in $B_H([t_o - h, t], E^n)$.

2.0 Problem statement

Consider a general initial value problem of system (1.1) of the form

$$\left.\begin{aligned} \dot{x}(t) &= f(t,\ x(t-h)), h > 0 \\ x(s) &= \varphi_0, t_o - h \le s \le t, \end{aligned}\right\} \tag{2.1}$$

where $\dot{x}(t)$ is derivative of the state function x(t) with respect to time *t*, and $x(t-h)$ is a continuous time lag function with $h > 0$ defining the delay interval. For a given initial condition $x(s) = \varphi_0,\ t_0 - h \le s \le t$, system (2.1) admits a unique solution.

***Theorem* 2.1**

Let $x(t)$ and f be continuous E^n–valued function with domain $D = \{(x : \|x(s) - \varphi_0(s)\| \prec h,\ s \in [t_0 - nh, t]\ n \ge 1)\}$ such that $f : (t_0 - nh, t) \to D$ is a contraction in $B_H([t_0 - h, t], E^n)$. Then there exists a unique solution of (2.1).

Proof

The integral equation equivalent of system (2.1) is given as

$$\varphi_{n+1}(t) = \varphi_n(t_0) + \int_{t_0 - h}^{t} f(s,\ \varphi_n(s))ds. \tag{2.2}$$

Assume $x(t) = \varphi_{n+1}(t)$ is a solution of (2.2) passing through $(\varphi_0(s), t_0)$, $(\varphi_{n+1}(s), t) \in B_H([t_0 - h, t], E^n) x E^n$ $([t_0 -)\ x\ E^n$ then $\dot{\varphi}_{n+1}(t) = f(t,\ \varphi_n(t-h))$. Since f is continuously differentiable on

the close interval $[t_0 - h, t]$, let there exists a positive real value $m > 0$ such that $f(t_1, \varphi(t-h)) > m$, for any $t_1 \in [t_0 - h, t]$. Suppose that at $t_1 \in [t_0 - h, t]$, $f(t_1, \varphi(t_0 - h)) = \varphi_1(s)$, where $t_0 - h \le s \le t_1$,
it implies

$$\varphi_1(s) = f(t_1, \varphi(t_0 - h)) > m \,. \tag{2.3}$$

Since $\varphi_1(s)$ is a solution on $[t_0 - h, t_1]$, then for every $t_2 > t_1$, $t_2 \in [t_0 - h, t]$, there exists a solution $\varphi_2(s) = f(t_2, \varphi(t_0 - h))$, where $t_0 - h \le s \le t_2$ and $\varphi_2(s) > \varphi_1(s)$. Therefore, for $t_n \in [t_0 - h, t]$, $\varphi_n(s) = f(t_n, \varphi(t_0 - h))$. Thus we have a sequence of nested solutions $\{ \varphi_n(s) \}$ in the close interval $[t_0 - h, t]$. Let f be a contraction in $B_H([t_0 - h, t], E^n) x E^n$. and for any real constant $0 < m_0 < 1$,

$$\| f(t, \varphi(t-h)) \| \le m_0 \| \varphi(s) \| \,, \; t_0 - h \le s \le t \tag{2.4}$$

holds. Thus $f(t_1, \varphi(t-h))$, $f(t_2, \varphi(t-h)), \cdots, f(t_n, \varphi(t-h))$ are closer for each $t_1 > t_2 > t_n$.

Assume $\{\varphi_n(s)\}$ to be a bounded monotone increasing sequence of solution, for any positive value $\varepsilon > 0$, there exists at least one positive integer $i > 0$ such that

$$\| \varphi(s) \| - \varepsilon < \varphi_i(s) \le \| \varphi(s) \| \tag{2.5}$$

Now, $\varphi_n(s) \le \varphi_{n+1}(s)$ for all n, hence for every n>1, $\varphi_n(s)$ satisfies (2.5), and hence

$$\| \varphi(s) \| - \varepsilon < \varphi_n(s) \le \| \varphi(s) \| + \varepsilon$$

The norm $\| \varphi(s) \| = \sup\limits_{t_0 - h \le s \le t} \| \varphi(s) \|$ defines the least upper bound of f and $\lim\limits_{n \to \infty} \varphi_n(s) = \varphi(s)$, $t_0 - h \le s \le t$. Also from (2.2),

$$\lim_{n\to\infty} \varphi_{n+1}(t) = \lim_{n\to\infty} \left(\varphi_n(t_0) + \int_{t_0 - h}^{t} f(s, \varphi_n(s))ds \right) = \lim_{n\to\infty} \varphi_n(t_0) + \int_{t_0 - h}^{t} \lim_{n\to\infty} (f(s, \varphi_n(s))ds)$$

$$= \lim_{n\to\infty} \varphi_n(t_0) + \int_{t_0 - h}^{t} \left(f(s, \lim_{n\to\infty} \varphi_n(s) \right) ds \; = \varphi(t) \tag{2.6}$$

Therefore by (2.5) and (2.6), solution of system (2.1) converge to the $\varphi(t)$, where $\varphi(t)$ is a continuous differentiable function on $B_H([t_0 - h, t], E^n)$.

Assuming $\varphi_{m+1}(t) = \varphi(t_0)_m + \int_{t_0 - h}^{t} f(s, \varphi_m(s))ds$ is another solution of (2.1) on $[t_0 - h, t]$ such that for any real value $k > 0$ on $[t_0 - h, t]$

$$\| \varphi_{n+1}(t) - \varphi_{m+1}(t) \| = \left\| \int_{t_0 - h}^{t} f(s, \varphi_n(s))ds - \int_{t_0 - h}^{t} f(s, \varphi_m(s))ds \right\| \le k \| V_0 \| \max_{t_0 - h \le s \le t} \| V(s) \| \,, \tag{2.7}$$

where $\| V_0 \| = \| t - (t_0 - h) \|$, and $\| V(s) \| = \| \varphi_n(s) - \varphi_m(s) \|$.

By the contraction of f in $B_H([t_o - h, t], E^n) \; x \; E^N$, k is the Lipschitz constant and f is bounded with a fixed point $\varphi(t)$. Thus (2.5) is the unique solution of (2.1).

3.0 Mathematical formulation

An analytic solution of (2.1) is not easily obtained unlike its equivalent ordinary differential system. A numerical method based on the concept of step-by-step approximation by varying the delay interval is formulated and solution is presented in form of a finite series.

The procedure involved the approximation of solution of the system for each T_i; $i = 1, 2, 3, \ldots, n$ delay subinterval as the delay h varies on a regular basis. The solution on the preceding interval is use to approximate the solution on the immediate succeeding interval, with the time function (t) depending on the origin of each subinterval under consideration Linearizing the time variants $\left(f(t,x(t-h))\right)$ of (2.0) with respect to *x(t)* results in a simple linear first order retarded equation represented by the initial value problem

$$\left.\begin{aligned} \dot{x}(t) &= a\ x\ (t-h), \quad h>0 \\ x_0(s) &= \varphi_0, \qquad t_0 - h \le s \le t, \end{aligned}\right\} \tag{3.1}$$

where 'a' is a scalar and $x(s)=\varphi_0$, is the initial value at $t_0 - h \le s \le t_0 \cdots$. System (3.1) admits a unique solution on $[t_0 - h, t)$. Consider (3.1) on T_i sub-interval,

$$\begin{aligned} &T_1: \quad t_0 - h \le t < t_0 \\ &T_2: \quad t_0 \le t < t_0 + h \\ &T_3: \quad t_0 + h \le t < t_0 + 2h \\ &T_4: \quad t_0 + 2h \le t < t_0 + 3h \\ &\vdots \\ &T_n: \quad t_0 + (n-1)h \le t < t_0 + nh,, \end{aligned}$$

for $h > 0$, the solution is formulated using a step-by-step concept of approximation for each delay subinterval as follows. By step-by-step approximation**,** considering (3.1) at $T_1 : t_0 - h \le t$ $< t_0$,

$$x_1(t) = a\int x_0(s)dt + c_1 = a\int \varphi_0 dt + c_1 = a\varphi_0 t + c_1 \tag{3.2}$$

At $t = t_0 - h$, $x_1(t) = \varphi_0$, and substituting for t and $x_1(t)$ in (3.2), we obtained

$$x_1(t) = \varphi_0 + a\varphi_0(t-(t_0-h)). \tag{3.3}$$

For $T_2; t_0 \le t < t_0 + h,$

$$x_2(t) = a\int x_1(t)dt + c_2 = a\int(\varphi_0 + a\varphi_0(t-(t_0-h)))dt + c_2$$

$$a\varphi_0 t + a^2\varphi_0(\frac{t^2}{2} - (t_0-h)t) + c_2. \tag{3.4a}$$

At $t = t_0$, $x_2(t) = \varphi_0 + a\varphi_0(t-(t_0-h))$, and substituting for t and $x_2(t)$ in (3.4a), we obtained,

$$x_2(t) = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0(\frac{t^2-t_0^2}{2!} - (t-t_0)(t_0-h)) \tag{3.4b}$$

For T_2; $t_0 + h \le t < t_0 + 2h$,

$$x_3(t) = a\int x_2(t)dt + c_3 = a\int(\varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0(\frac{t^2-t_0^2}{2!} - (t-t_0)(t_0-h)))dt + c_3. \tag{3.5a}$$

At $t = t_0 - h$, x_3, $\quad (t) = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0(\frac{t^2-t_0^2}{2!} - (t-t_0)(t_0-h)),$

substituting for t and x_3(t) in (3.5a) we obtained

$$\begin{aligned} x_3(t) &= \varphi_0 + a\varphi_0\left((t-(t_0+h)\right) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} + (t-(t_0+h)t_0(t_0-h))\right) \\ &+ a^3\varphi_0(\frac{t^3-(t_0+h)^3}{3!} - (\frac{t^2-(t_0+h)^2}{2!} - (t-(t_0+h)t_0)(t_0-h))). \end{aligned} \tag{3.5b}$$

For $T_{n;}$ $t_0 + (n-1)h \le t < t_0 + nh$,

$$x_n(t) = a\int x_{n-1}(t)dt + c_{n-1}$$

$$= a\int ((\varphi_0 + a\varphi_0((t-(t_0-h)))) + a^2\varphi_0\left(\frac{t^2-t_0{}^2}{2!}\right) + (t-(t_0-h))t_0(t_0-h))dt +$$

$$+ a\int\left(\left(a^3\varphi_0\left(\frac{t^3-(t-h)^3}{3!} - \frac{t^2-(t_0+h)^2}{2!} - (t-(t_0+h)t_0)(t_0-h)\right)\right)\right)dt \ \ldots$$

$$+ a\int\left(a^{n-1}\varphi_0\left(\frac{t^{n-1}-(t_0+(n-1)h)^{n-1}}{(n-1)!} \ldots (t-(t_0-(n-1)h)\ldots t_0(t_0-h)\right)\right)dt + c_n$$

At $t = t_0 - (n-1)h, \ \ x_n(t) = x_{n-1}(t), ,$

$$x_n = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} - (t-(t_0+h)t_0(t_0-h))\right)$$

$$+ a^3\varphi_0\left(\frac{(t^3-(t_0+h)^3}{3!} + \frac{(t^2-(t_0+2h)^2}{2!} - (t-(t_0+h)t_0(t_0-h)\right) + \ldots$$

$$+ \ a^{n-1}\varphi_0\left(\frac{(t^{n-1}-(t_0+(n-1)h)^{n-1}}{(n-1)!} - \cdots - (t-(t_0-(n-1)h)\ \ldots t_0(t_0-h)\right) \qquad (3.6)$$

$$+ a^n\varphi_0\left(\frac{t^n-(t_0+nh)^n}{n!} + \frac{t^{n-1}-(t_0+(n-1)h)^{n-1}}{n!} - \cdots - (t-(t_0+nh)(t_0-(n-1)h)\ldots(t_0-h)\right)$$

4.0 Stability analysis

Considering the resulting approximate solution of system (3.1) on each T_i; $i = 1, 2, 3, \ldots n$ delay subinterval with a corresponding initial condition as stated below,

$T_1: \quad t_0 - h \le t < t_0$ and $x_0(t_0 - h) = \varphi_0$,

$x_1(t) = \varphi_0 + a\varphi_0(t-(t_0-h)).$

$T_2: t_0 \le t < t_0 + h$, and $x_1(t_0) = \varphi_0 + a\varphi_0(t-(t_0-h))$,

$x_2(t) = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0(\frac{t^2-t_0^2}{2!} - (t-t_0)(t_0-h))$ $T_3: t_0 + h \le t < t_0 + 2h,$

and $\quad x_2(t_0+h) = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0(\frac{t^2-t_0^2}{2!} - (t-t_0)(t_0-h))$,

$$x_3(t) = \varphi_0 + a\varphi_0((t-(t_0+h)) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} + (t-(t_0+h)t_0(t_0-h))\right)$$

$$+ \ a^3\varphi_0(\frac{t^3-(t_0+h)^3}{3!} - (\frac{t^2-(t_0+h)^2}{2!} - (t-(t_0+h)t_0)(t_0-h))).$$

$T_n;\ t_0 + (n-1)h \le t < t_0 + nh$, and $x_n(t_0 + (n-1)h) = x_{n-1}(t)$,

$$x_n = \varphi_0 + a\varphi_0(t-(t_0-h)) + a^2\varphi_0\left(\frac{t^2-t_0^2}{2!} - (t-(t_0+h)t_0(t_0-h))\right)$$

$$+a^3\varphi_0\left(\frac{(t^3-(t_0+h)^3}{3!} + \frac{(t^2-(t_0+2h)^2}{2!} - (t-(t_0+h)t_0(t_0-h)\right) + \ldots$$

$$+\ a^{n-1}\varphi_0\left(\frac{(t^{n-1}-(t_0+(n-1)h)^{n-1}}{(n-1)!} - - - - (t-(t_0-(n-1)h)\ \ldots t_0(t_0-h)\right)$$

$$+a^n\varphi_0\left(\frac{t^n-(t_0+nh)^n}{n!} + \frac{t^{n-1}-(t_0+(n-1)h)^{n-1}}{n!} - - - (t-(t_0+nh)(t_0-(n-1)h)\ldots(t_0-h)\right)$$

The asymptotic stability properties will now be analyzed for any change in the initial condition $x(s)=\varphi_0,\ t_0-h\le s\le t$.

***Definition* 4.1** Han [8]

(i) The solution *x(t)* of system (3.1) is Lyapunov stable if for any $\varepsilon > 0$, there exists $\delta = \delta(t-h,\ \varepsilon) > 0$ such that if $\|\varphi(s)\| < \delta$ then $\|x(t-nh,\varphi(s))\| < \varepsilon,\ t_0-h\le s\le t_0$.

(ii) The solution *x(t)* of (3.1) is asymptotically stable if it is Lyapunov stable, and there exists a $\delta_1 = \delta_1(t-h)$ satisfying $\|\varphi(s)\| < \delta_1$ such that $\|x(t-nh,\varphi(s))\| \to 0$ as $t \to \infty$.

(iii) The solution *x(t)* is uniformly asymptotically stable if it is stable, and furthermore there exists $\delta_2 > 0$ (independent of *t-h*) such that if $\|\varphi(s)\| < \delta_2$, then $\|x(t-nh,\varphi(s))\| \to 0$ as $t\to\infty$.

***Theorem* 4.1**

Assume $f : B_H\ x\ D \to E^n$ for $D\subset E^n$ is continuous, satisfy local Lipschitzian condition on $[t_0+nh, t],\ n=0,1,2\ldots$, and global Lipschitzian condition on $[t_0-h,\ \infty)$,and is compact in D . The resulted solution $x(\,t-nh,\ \varphi_0(s))$ of (3.1) is,

(i) Lyapunov stable if for any change in the initial condition $x(t)=\varphi_0(s),\ t_0-h\le s\le t_0$, the solution $x(\,t-nh,\ \varphi_0(s))$ remained valid on the entire $[t_0-h,\ \infty)$.

(ii) Asymptotically stable if limit $\|x(t-nh,\ \varphi_0(s))\| = 0$ for an infinite increment in time (*t*).

Proof

Since (3.1) is continuous on each $[\ t_0+nh, t]$ for $n=0,1,\ 2,\ldots$ and by the approximating technique formulation, let there exist solutions,

$$x(t) = f\left(\,t-nh,\ \varphi_0(s)\right) \text{ and } \bar{x}(t) = f\left(\,t-nh,\ \bar{\varphi}_0(s)\right), \text{ for } t_0-h\le s\le t_0. \qquad (4.1)$$

satisfying (3.1). If for any pre-determined constant $\varepsilon > 0$ there exists a $\delta = \delta(t-h,\ \varepsilon)$, for $0\le t-h<\delta<\varepsilon$, such that $\|x(t)-\bar{x}(t)\| < \delta$ holds, it follows that,

$$\left\| f\left(t-h,\ \varphi_0(s)\right) - f\left(t-h,\ \bar{\varphi}_0(s)\right)\right\| < \varepsilon\,. \qquad (4.2)$$

This implies that $x(t-nh,\ \varphi_0(s))$ is valid on $[t_0-h,\ \infty)$, and is Lyapunov stable.

If δ_1 is a constant, and $0\le t-h<\delta_1<\varepsilon$ such that $\delta_1=\delta_1(t-h)$>0, then (4.1) implies $\|x(t)-\bar{x}(t)\| < \delta_1$. Also since (3.1) is locally Lipschitzian on $[t_0+nh, t]$, then $x_i(\,t-nh,\ \varphi_0(s)),\ i=1,\ 2,\ 3,\ \ldots n$ of (3.3), (3.4) and (3.6) are monotone functional solutions on the bounded interval $[t_0+(n-1)h,\ t_0+nh]$.

By Weierstrass–Bolzano concept of boundeness in a close interval,

$$\left\|x_1(t_0+h,\varphi_0(s))-\bar{x}_1(t_0+h,\bar{\varphi}_0(s))\right\|=\left\|\frac{x(t_0+h,\varphi_0(s))-\bar{x}(t_0+h,\bar{\varphi}_0)(s))}{2}\right\|,$$

$$\left\|x_2(t_0+2h,\varphi_0(s))-\bar{x}_2(t_0+2h,\bar{\varphi}_0(s))\right\|=\left\|\frac{x(t_0+h,\varphi_0(s))-\bar{x}(t_0+h,\bar{\varphi}_0(s))}{2^2}\right\|,$$

$$\vdots$$

$$\left\|x_n(t_0+nh,\varphi_0(s))-\bar{x}_n(t_0+nh,\bar{\varphi}_0(s))\right\|=\left\|\frac{x(t_0+h,\varphi_0(s))-\bar{x}(t_0+h,\bar{\varphi}_0(s))}{2^n}\right\|,$$

holds. Therefore,

$$\begin{aligned}\left\|x(t_0+h,\varphi_0(s))-\bar{x}(t_0+h,\bar{\varphi}_0(s))\right\| &\le \left\|x_1(t_0+h,\varphi_0(s))-\bar{x}_1(t_0+h,\bar{\varphi}_0(s))\right\| \\ &+\left\|x_2(t_0+2h,\varphi_0(s))-\bar{x}_2(t_0+2h,\bar{\varphi}_0(s))\right\|+\left\|x_n(t_0+nh,\varphi_0(s))-\bar{x}_n(t_0+nh,\varphi_0(s))\right\| \\ &=\left\|\frac{x(t_0+h,\varphi_0(s))-\bar{x}(t_0+h,\bar{\varphi}_0(s))}{2^n}\right\|. \qquad (4.3)\end{aligned}$$

By the continuity of f on $[t_0+nh,t]$, the solutions $x_n(t-h,\varphi_0(s))$ and $\bar{x}_n(t-h,\varphi\ (s))$ converge to $\varphi(t)$ *and* $\bar{\varphi}(t)$ respectively. Thus f forms a compact set in *D*, and

$$\lim_{\substack{n\to\infty\\ t\to\infty}}\left\|\frac{\varphi(t)-\bar{\varphi}(t)}{2^n}\right\|=0. \qquad (4.4)$$

Therefore the resulted solution of (3.1) is asymptotically stable.

5.0 Illustration

Consider the retarded system of the form,

$$\begin{aligned}&\dot{x}(t)=1+x(t-1)\\ &x_0(1)=1,\ x_0(t-1)=1\ \textit{and}\quad t\in[1,\infty)\end{aligned} \qquad (5.1)$$

By step-by-step approximation, $x_1(t)$ on $1\le t<2$,

$$x_1(t)=\int\ (1+x_0(t-1))dt+c_1=\int 2dt+c_1 \qquad (5.2)$$

Solving (5.2) at an initial state of $x_0(1)=1$,

$$x_1(t)=2t-1\ =1-2[-(t-1)], \qquad (5.3a)$$

and

$$x_1(t-1)=1-2[-(t-2)].. \qquad (5.3b)$$

Considering $x_2(t)$ on $2\le t<3$,

$$x_2(t)=\int\ (1+x_1(t-1))dt+c_2\ =\int\ (1+2(t-2)+1)dt+c_2=t^2-2t+c_2 \qquad (5.4)$$

Solving (5.4) at an initial state of $x_1(2)=1-2[-(t-1)]$,

$$x_2(t)=t^2-2t+3=1-2\left[-(t-1)-\frac{(t-2)^2}{2!}\right], \qquad (5.5a)$$

and $$x_2(t-1) = 1-2\left[-(t-2)-\frac{(t-3)^2}{2!}\right]. \qquad (5.5b)$$

Considering $x_3(t)$ on $3 \le t < 4$,

$$x_3(t) = \int (1+x_2(t-1))dt + c_3 = \int (1+2\left[(t-2)+\frac{(t-3)^2}{2!}\right]+1)dt + c_3 = \int (t^2-4t+7)dt + c_3 \qquad (5.6)$$

Solving (5.6) at an initial state of $x_2(3) = 1-2\left[-(t-1)-\frac{(t-2)^2}{2!}\right]$,

$$x_3(t) = \frac{t^3}{3} - 2t^2 + 7t - 9 = 1-2\left[-(t-1)-\frac{(t-2)^2}{2!}-\frac{(t-3)^3}{3!}\right], \qquad (5.7a)$$

and $$x_3(t-1) = 1-2\left[-(t-2)-\frac{(t-3)^2}{2!}-\frac{(t-4)^3}{3!}\right]. \qquad (5.7b)$$

Therefore for $x_n(t)$ on $n \le t < n+1$

$$x_n(t) = \int (1+x_{n-1}(t-1))dt + c_n$$

$$x_n(t) = 1-2\left[-(t-1)-\frac{(t-2)^2}{2!}-\frac{(t-3)^3}{3!}-\frac{(t-4)^4}{4!}-\ldots-\frac{(t-n)^n}{n!}\right],$$

and $$x_n(t-1) = 1-2\left[-(t-2)-\frac{(t-3)^2}{2!}-\frac{(t-4)^3}{3!}-\frac{(t-5)^4}{4!}-\ldots-\frac{\left(t(n+1)\right)^n}{n!}\right].$$

Indeed, the general solution of (5.1) is expressed as

$$x_n(t) = 1-2\exp\left(-(t-i)\right), 1 \le i \le n, \qquad (5.8a)$$

where *i* measures the change in time lag (*h*) and

$$x_n(t-1) = 1-2\exp\left(-(t-(i+1))\right). \qquad (5.8b)$$

The result (5.8a,b) above is comparative to the iterative method for an equivalent ordinary differential equation of (5.1)

Also by Theorem 4.1, the solution *x(t)* is asymptotically stable, if $\|x_1(t)-x_0(t)\| < \delta$ such that $\lim_{t\to\infty}\|x_1(t)-x_0(t)\| = 0$. Using solutions (5.3a), (5.5a) and (5.7a) with the initial conditions, then

$$\lim_{t\to\infty}\|x_{n+1}(t)-x_n(t)\| = \lim_{t\to\infty} 2\exp(-(t-i)) = 0$$

This implies that every solution of system 5.1 is asymptotically stable.

6.0 Conclusion

The theorem on the existence and uniqueness of solution of delay retarded system is established and proved using the continuity and Lipschitzian conditions. An approximate solution of the system for each n-subinterval and the asymptotic stability property is analyzed. Results obtained are comparable to general solution form of the ordinary differential system.

References

[1] Asl, F. M. and Ulsoy A. G. (2003). Analysis of System of Linear Delay Differential Equations, Journal of Dynamic Systems, Measurement and Control, (125), 215-223.

[2] Cheban D. N. (2002). Uniform Exponential Stability of Linear Almost Periodic Systems in Banach Spaces, Electronic Journal of Differential Equation, (20), 29.

[3] Davies, I. (2006). Criteria for Stability and Boundedness of Linear Delay System, Journal of the Mathematical Association of Nigeria, (33) no 2B, 419-426.

[4] Driver, R. D. (1977). Ordinary and Delay Differential Equations, Sprmger-Verlag, New York.

[5] Falbo, C. E. (1995). Analytic and Numerical Solutions to Delay Differential Equations, Joint Meeting of Northern and Southern California Sections of the MAA.
[6] Hale, J. K. (1977). Theory of Differential Equation, Sprnger-Verlag, New-York.
[7] Hale, J. K. and Cruz, M. A. (1970). Existence, Uniqueness and Continuous Dependence for Hereditary Systems, Annali di Mathematica Pura ed Applicata (85), 63-82.
[8] Han, Q. L. (2001). On Delay-Dependent Stability for Neutral Delay Differential Systems (II) no 4, 965-976.
[9] Hmamed, A. (1986). Stability Conditions of Delay-Differential System, Int. J. Control, (43), no 2, 455-463.
[10] Kreyszig, E. (1979). Advanced Engineering Mathematics, John Wiley and Sons, New York.
[11] Lam, J. (1991). Balanced Realization of Pade Approximation of e^{-ST} (All-pass Case), IEEE Trans. Autom Control (39), 1096-1110.
[12] Liu, X. Y. and Mansour, M. (1984). Stability Test and Stability Conditions for Delay Differential Systems, Int. J. Control (39), no 6, 1229-1242.
[13] Onwuatu, J. U. (1993). Null Controllability of Non-Linear Infinite Neutral-System, KYBERNE TIKA (29) no x, 484-487.