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Thermally radiating fluid: Approximations of integral solutions

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Abstract

This paper examines thermally radiating fluid. Integral solutions are presented which are evaluated numerically. A new and simpler approach to the approximate form of the integral solutions is presented that gives rise to approximate analytical solutions. It is shown that the results reveal the characteristics of the problem and compare favourably well.

Keywords
Integral solutions, numerical integrations; optically thin incompressible fluid; thermally radiating fluid

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1.0 Introduction

The study of thermal radiation, commonly known as radiation heat transfer, occurs in many engineering applications [1]. A distinguishing feature of radiation heat transfer is that it is associated with the radiation heat flux, which is proportional to the differences of individual absolute temperatures of the bodies each raised to the fourth power. Consequently, the importance of radiation becomes intensified at high temperatures. For example, high temperature phenomena or high-power radiation sources are common in solar physics-particularly in astrophysical studies [2], in combustion applications such as fires, furnaces, IC engines, in nuclear reactions such as in the sun or in nuclear explosions [3], in compressors in ships and in gas flares from petrochemical industry [4]. For air, the contribution of radiation becomes significant when the wall temperature is in the range $6000-10,000K$. This situation is encountered for re-entry space vehicles. Radiation effect is also important for nitrogen-gas-soot mixtures including H_2O, CO_2, CO, CH_4, NO, SO_2, N_2O, NH_1 and C_2H_2 in the temperature range $300-3000K$ [5].

A primary difficulty in modelling radiation heat transfer problem is the involvement of a nonlinear integro-differential equation of the radiation heat flux in the governing energy equation. This aspect of radiation heat transfer is unique and requires a special computational treatment. At best numerical computations are formulated to tackle such an equation, or on the other hand, fairly realistic assumptions are made in order to proffer approximate analytical solutions. The objective of the present paper is the consideration of integral solutions, which are in turn evaluated numerically. A new and simpler approach to the approximate form of the integral solutions is presented that give rise to approximate analytical solutions. The present considerations in radiation heat transfer studies have great import, thereby widening the applicability of the results.

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Nomenclature

ρ - fluid density
ν - kinematic viscosity
T - temperature of fluid
T_∞ - free stream temperature
T_w - wall temperature
Θ - dimensionless temperature
c - specific heat at constant pressure
y - transverse co-ordinate
q - radiation flux
κ - thermal conductivity
σ - Stefan-Boltzmann constant
α - absorption coefficient or penetration depth
$\Pr$ - Prandtl number
N - radiation parameter

In Section 2, the mathematical formulations of the problem and the non-dimensional form of the governing equations are established. Solutions to these equations are obtained in Section 3. The results of the previous sections are discussed in Section 4. In Section 5, general concluding remarks of the results of the previous sections are given.

2.0 Mathematical formulations

The physical problem consists of an optically thin incompressible thermally radiating fluid near a vertical infinite plate. In some respects, the physics of radiation does not require details of the flow field at the radiating surface, and that we need only be concerned with the surface itself, as well as any other surfaces that are radiating to the surface of interest [6]. The steady state governing energy equation with constant viscosity ν and thermal conductivity κ that incorporates radiation heat flux and the general radiation heat flux equation with constant absorption or penetration depth α in one space coordinate y as in Cheng [7] are, respectively given as follows: $\frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c}\frac{\partial q}{\partial y} = 0$,

(2.1)

$$\frac{\partial^2 q}{\partial y^2} - 3\alpha^2 q - 16\sigma\alpha T^3 \frac{\partial T}{\partial y} = 0. \qquad (2.2)$$

From equation (2.2), four different limits may be considered depending on the absorption coefficient α. Thus, we define $\alpha << 1$ as optically thin and $\alpha >> 1$ as optically thick. The limiting case $\alpha = 0$ represents a non-participating medium (transparent) where the radiation flux is constant $\left(\frac{\partial q}{\partial y} = 0\right)$, whereas $\alpha = \infty$ corresponds to an opaque medium in which $q = 0$.

An example for condition of an optically thin environment is found in the intergalactic layers where the plasma gas is assumed to be of low density [6]. In this case equation (2.2) becomes

$$\frac{\partial q}{\partial y} = 4\alpha\sigma\left(T^4 - T_\infty^4\right). \qquad (2.3)$$

Furthermore, when it is assumed that the differences within the fluid are sufficiently small, then T^4 can be expressed as a linear function of temperature in Taylor series about T_∞ neglecting higher order terms. Thus, $T^4 \cong 4T_\infty^3 T - 3T_\infty^4$. (2.4)

Equation (2.3) is now written as

$$\frac{\partial q}{\partial y} = 16\alpha\sigma T_\infty^3\left(T - T_\infty\right) \qquad (2.5)$$

This is known as the linear differential approximation of Cogley-Vincenti-Gilles equilibrium model [8] of the radiation flux.

In general the optically thin boundary layer is a physically realistic model, however, it is worth mentioning that an optically thick model may be used if the thermal layer has become very thick or the medium is highly absorbing. This is otherwise known as Rosseland approximation. Therefore, it follows from equation (2.2) that

$$\frac{\partial q}{\partial y} = -\frac{4\sigma}{3\alpha}\frac{\partial^2 T^4}{\partial y^2}. \qquad (2.6)$$

For the purpose of this study, we shall only consider the optically thin condition. The application of equations (2.3) and (2.5) to equation (2.1) gives respectively

$$\frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial y^2}-\frac{4\sigma\alpha}{\rho c}\left(T^4-T_\infty^4\right)=0 \qquad (2.7)$$

and

$$\frac{\kappa}{\rho c}\frac{\partial^2 T}{\partial y^2}-\frac{16}{\rho c}T_\infty^3\left(T-T_\infty\right)=0. \qquad (2.8)$$

The associated boundary conditions are given by $T=T_w$ at $y=0$, (2.9)

and $T=T_\infty$ as $y\to\infty$. (2.10)

In order to facilitate the analysis, it is now convenient to introduce the following non-dimensional quantities: $Y=\frac{1}{L}y,\ \Theta=\frac{T}{T_\infty},\ \Theta_w=\frac{T_w}{T_\infty},\ N=\frac{4\sigma\alpha L^2}{\rho\nu c}T_\infty^3,\ \Pr=\frac{\rho\nu c}{\kappa},$

where L is a characteristic length of the plate. The governing equations (2.7) and (2.8) are now written in non-dimensional form as

$$\frac{\partial^2\Theta}{\partial Y^2}-\Pr N\left(\Theta^4-1\right)=0, \qquad (2.11)$$

$$\frac{\partial^2\Theta}{\partial Y^2}-4\Pr N\left(\Theta-1\right) \qquad (2.12)$$

The boundary conditions (2.9) – (2.10) are written as

$$\Theta=\Theta_w \text{ at } Y=0, \qquad (2.13)$$

$$\Theta=1 \text{ as } Y\to\infty \qquad (2.14)$$

Here the parameters entering the problem are N, radiation parameter and $\Pr$, Prandtl number. The mathematical formulations of the problem are now complete.

3.0 Method of solution

Here solutions are advanced for equations (2.11 – 2.12) with the aid of the equations (2.13) – (2.14). Multiplying both sides of equation (2.11) by $2\left(\partial\Theta/\Theta Y\right)$ and integrating with the aid of the boundary conditions, the following integral solution [2, 9] is obtained

$$Y=\left(\frac{5}{2\Pr N}\right)^{1/2}\int_{\Theta}^{\Theta_w}\frac{d\varsigma}{\sqrt{\zeta^5-5\zeta+4}}. \qquad (3.1)$$

A similar solution is hereby deduced to the linear equation (2.12) as

$$Y=\frac{1}{2\sqrt{\Pr N}}\int_{\Theta}^{\Theta_w}\frac{d\varsigma}{\sqrt{\zeta^2-2\zeta+1}} \qquad (3.2)$$

The results (3.1) and (3.2) are physically meaningful for numerical integrations. An exact solution to equation (2.12) is hereby deduced as follows:

$$\Theta=\left(\Theta_w-1\right)\exp\left(-2\sqrt{\Pr N}Y\right)+1. \qquad (3.3)$$

It is inferred from the exact solution (3.3) that in the presence of an intense radiation (i. e. $N\to\infty$), $\Theta\to1$. This is a limiting value, where all other values of Θ due to $N<\infty$ asymptotically approach. Figure 3.1 clearly depicts this situation.

The main results of the investigation are herein considered. From Abramowitz and Stegun [10], if $b<<a$, then the approximate value

$$\left(a+b\right)^k\approx a^k+ka^{k-1}b \qquad (3.4)$$

holds. It is seen from equation (3.1) that $-5\zeta+4<<\zeta^5$ provided $\zeta\geq1$. Therefore, the approximate relation

$$\left(\zeta^5 - 5\zeta + 4\right)^{-1/2} \approx \frac{1}{\zeta^{5/2}} - \frac{2}{\zeta^{15/2}} + \frac{5}{2\zeta^{13/2}} \qquad (3.5)$$

is obtained according to equation (3.4). Consequently, an implicit analytical solution in Θ given by

$$Y = \frac{1}{429}\left(\frac{5}{2\Pr N}\right)^{1/2}\left(-\frac{-132+195\Theta_w+286\Theta_w^5}{\Theta_w^{13/2}} - \frac{-132+195\Theta+286\Theta^5}{\Theta^{13/2}}\right) \qquad (3.6)$$

is derived from the result (3.1).

Similarly, for $\zeta \geq 1$, it is inferred from equation (3.2) that $-2\zeta + 1 << \zeta^2$. Therefore, the relation

$$\left(\zeta^2 - 2\zeta + 1\right)^{-1/2} \approx \frac{1}{\zeta} - \frac{1}{2\zeta^3} + \frac{1}{\zeta^2} \qquad (3.7)$$

is another valid approximation according to equation (3.4). In this case, the implicit analytical solution can be written as

$$Y = \left(\frac{1}{2\Pr N}\right)^{1/2}\left(\ln(\Theta_w) + \frac{1}{4\Theta_w^2} - \frac{1}{\Theta_w} - \ln(\Theta) + \frac{1}{4\Theta^2} - \frac{1}{\Theta}\right) \qquad (3.8)$$

The new approximate analytical solutions (3.7) and (3.8) are used to validate respectively the integral solutions (3.1) and (3.2) using numerical integrations.

4.0 Discussion of results

The problem of thermally radiating and optically thin incompressible fluid past a vertical infinite plate has been solved analytically and numerically. Firstly, integral solutions are obtained for the resulting nonlinear and linear steady-state energy equations. Secondly, approximate analytical solutions (3.7) and (3.8) are used to validate the integral solutions (3.1) and (3.2), which are evaluated using Simpson's rule with a double precision arithmetic (40 digits is used in the computations with **MAPLE** package in a Macintosh Pentium 4 Machine). Tables 5.1 and 5.2 display the computations and typical parameter values used for the computations are indicated. In particular, as the Prandtl number $\Pr$ is a measure of the relative importance of the viscosity and thermal diffusivity of the fluid, it is set equal to a fixed value of 0.71 throughout the investigations, which physically corresponds to an astrophysical body (air) at 20^0C.

Table 5.1: Comparison of numerical and approximate values for equations (3.1) and (3.6) at variations of radiation parameter with $\Pr = 0.71$ and $\square_w = 2.4$

	***N* = 0.2**			***N* = 0.5**			***N* = 1.0**		
Y	**Num.**	**Approx.**	**R. Error**	**Num.**	**Approx.**	**R. Error**	**Num.**	**Approx.**	**R. Error**
0.0	2.4000	2.4000	0.0000	2.4000	2.4000	0.0000	2.4000	2.4000	0.1149
0.2	2.0709	2.0690	0.0019	1.9300	1.9265	0.0035	1.8007	0.0000	0.9298
0.4	1.8442	1.8395	0.0047	1.6563	1.6468	0.0095	1.5050	1.7951	0.1476
0.6	1.6790	1.6702	0.0088	1.4793	1.4606	0.0187	1.3340	0.0056	0.8720
0.8	1.5539	1.5398	0.0141	1.3577	1.3265	0.0312	1.2269	1.4880	0.1835
1.0	1.4566	1.4359	0.0207	1.2708	1.2244	0.0466	1.1567	0.0170	0.8162
1.2	1.3793	1.3509	0.0284	1.2071	1.1433	0.0638	1.1096	1.2993	0.2243
1.4	1.3172	1.2798	0.0374	1.1596	1.0764	0.0834	1.0774	0.0347	0.7433
1.6	1.2665	1.2191	0.0474	1.1237	1.0196	0.1041	1.0555	1.1692	0.2871
1.8	1.2248	1.1665	0.0583	1.0964	0.9700	0.1264	1.0405	0.0577	0.7088
2.0	1.1902	1.1202	0.0700	1.0756	0.9255	0.1501	1.0304	1.0721	0.3167
2.2	1.1614	1.0790	0.0834	1.0596	0.8845	0.1751	1.0235	0.0846	0.6962
2.4	1.1373	1.0419	0.0954	1.0475	0.8451	0.2024	1.0187	0.9947	0.3225

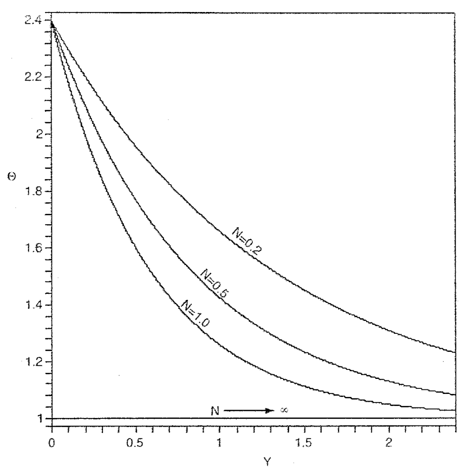

Figure 5.1: Temperature profiles (exact solution, equation (3.3)) as a function of *Y* for variations in the radiation parameter with Pr = 0.7 and θ_w = 2.4

Table 5.2: Comparison of exact, numerical and approximate values for equations (3.3), (3.2) and (3.8) at variations of radiation parameter with Pr = 0.7 and θ_w = 2.4.

Y	***N* = 0.2**				***N* = 0.5**			
	Exact	**Num.**	**Approx.**	**R. Error**	**Exact**	**Num.**	**Approx.**	**R. Error**
0.0	2.4000	2.4000	2.4000	0.0000	2.4000	2.4000	2.4000	0.0000
0.2	2.2041	2.2041	2.2164	0.0123	2.1031	2.1031	3.0615	0.9584
0.4	2.0356	2.0356	0.1833	1.8523	1.8692	1.8692	2.6860	0.8168
0.6	1.8907	1.8907	0.1857	1.7050	1.6849	1.6849	2.3637	0.6788
0.8	1.7661	1.7661	0.1882	1.5779	1.5396	1.5396	2.0860	0.5464
1.0	1.6589	1.6589	0.1908	1.4681	1.4252	1.4252	0.1865	1.2387
1.2	1.5667	1.5667	0.1936	1.3731	1.3350	1.3350	0.1906	1.1444
1.4	1.4874	1.4874	0.1966	1.6774	1.2640	1.2640	0.1951	1.0689
1.6	1.4192	1.4192	0.1997	1.2195	1.2080	1.2080	0.2000	1.0080
1.8	1.3506	1.3506	0.2031	1.1475	1.1639	1.1639	0.2055	0.9584
2.0	1.3101	1.3101	0.2067	1.1034	1.1291	1.1291	0.2117	0.9174
2.2	1.2667	1.2667	0.2106	1.0561	1.1018	1.1018	0.9253	0.1765
2.4	1.2294	1.2294	0.9823	0.2471	1.0802	1.0802	0.8263	0.2539

Table 5.2: Comparison of exact, numerical and approximate values for equations (3.3), (3.2) and (3.8) at variations of radiation parameter with Pr = 0.7 and θ_w = 2.4. (continued)

Y	***N* = 1.0**			
	Exact	**Num.**	**Approx.**	**R. Error**
0.0	2.4000	2.4000	2.4000	0.0000
0.2	1.9994	1.9994	2.0120	0.0126
0.4	1.7135	1.7135	0.1894	1.5241
0.6	1.5093	1.5093	0.1957	1.3136
0.8	1.3636	1.3636	0.2029	1.1607
1.0	1.2596	1.2596	0.2114	1.0482
1.2	1.1853	1.1853	0.8881	0.2972
1.4	1.1323	1.1323	0.7555	0.3768
1.6	1.0944	1.0944	0.6372	0.4572
1.8	1.0874	1.0874	0.5247	0.5627
2.0	1.0481	1.0481	0.3646	0.6835
2.2	1.0344	1.0344	0.3329	0.7015
2.4	1.0245	1.0245	0.3055	0.7190

From the exact result (3.3), it is observed that increasing the radiation parameter decreases the temperature exponentially (see Figure 5.1). A salient feature is the presence of an asymptocity. The curves approach the abscissa asymptotically implying that $\Theta = 1$ is a limiting case. This is the value due to an intense radiation (i. e. $N \to \infty$).

The results shown in Table 5.1 compare the numerical evaluation of equation (3.1) and the approximate analytical solution (3.6) with different values of the radiation parameter N for $0 \le Y \le 2.4$. It is observed that the approximate analytical solution is relatively consistent in accordance with the numerical integration as the relative errors (i. e. R. Error) revealed. Furthermore, it is seen that the relative errors increase with an increasing radiation parameter as Y increases. It is observed that $\Theta \to 1$ for large values of N from the numerical calculations. For example, putting N = 2000, the value of □→1.000121055650152228005933309826897266
675 (40 digits) for all values of $0 \le Y \le 2.4$. Consequently, $\Theta = 1$ is a limiting value as $N \to \infty$. This is observed from the exact result (see Figure 1).

Table 5.2 compare the results of exact, numerical and approximate values of equations (3.3), (3.2) and (3.8) at different values of N. It is seen that the numerical integration preserves the exact structure, and compares favourably well with the approximate analytical solution. Once again, it is observed that for large values of $N, \Theta \to 1$. Specifically, putting $N = 2000$ in the numerical integration of equation (3.2) gives $\Theta \to 1.00012105565015222800593330\quad 9826897266675$ (40 digits) for all values of $0 \le Y \le 2.4$. This implies that the two integral solutions (3.1) and (3.2) for Θ converges to 1, which is the value as $N \to \infty$.

From the numerical integrations (3.1) (see Table 5.1) and (3.2) (see Table 5.2), it is observed that the approximate analytical solution (3.6) due to the approximate relation (3.5) gives a better result to the equation (2.11) than the equation (2.12) due to the linear approximation of the nonlinear term of the fluid temperature in equation (2.3). In any case, the solution due to the linearization is a good tool for testing and validating numerical schemes of the equation (2.11).

5.0 Conclusion

The problem of thermally radiating and optically thin incompressible fluid past a vertical infinite plate has been examined. Integral, analytical and numerical solutions are presented. The effect of radiation transfer has been investigated. It is observed that increase in radiation reduces the temperature.

It is primarily observed that the values of $\Theta \to 1$ for large values of the radiation parameter. This limiting value of Θ suggests a boundary layer character of the steady-state nature of the temperature field. It is evident that the approximate analytical solutions reveal the characteristics of the problem and compare favourably well with the numerical integrations.

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