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# Analysis of temperature distribution in a heat conducting fiber with convection losses using finite element method

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***Abstract***

***The temperature distribution in a heat conducting fiber is computed using the Galerkin Finite Element Method in the present study. The weak form of the governing differential equation is obtained and nodal temperatures for linear and quadratic interpolation functions for different mesh densities are calculated for Neumann boundary conditions. The results show that using a mesh of three quadratic elements produces a maximum error of* 0.622 *compared to* 1.1832 *for a similar number of linear elements. It is concluded that as the mesh is refined further progressively, the finite element solution approaches the exact solution admirably. The results are displayed in both graphical and tabular forms.***

**Nomenclature**

$a_j$: nodal degree of freedom of $j^{th}$ node
$[f^e]$: Internal load (source vector) of element
$H$: Convective heat transfer coefficient
$[K]$: Element characteristic matrix
$K$: Thermal conductivity
$H$: Length of element
$L$: Length of fiber
$N$: Number of degree of freedom
$Q_i^e$: Boundary flux of $i^{th}$ node of $e^{th}$ element
$Q$: Heat flux
$R$: Radius of the fiber
$T$: Temperature ($^oF$)
$T_m$: Temperature of fluid in which fiber is immersed
$x_a$: $x$-coordinate of left node of element
$x_b$: $x$-coordinate of right node of element
$Ø$: Shape functions for linear and quadratic interpolation
$\Delta$: Kronecker delta

## 1.0 Introduction

The thermal performance of fibers used in telecommunication systems is of concern to both the circuit and Physical design Engineer Burnett, [1]. Heat generated in some of the components (such as transistors and resistors) if not dissipated rapidly enough to the surrounding air can significantly impair performance. It is therefore important to be able to determine the temperature distribution in fibers while in service so as to design means of dissipating the heat to the surroundings

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Different methods of solution exist for solving equations governing heat conduction in Engineering and Science. These include the finite element, finite difference and the finite volume methods respectively (William, [8]). The finite difference and finite volume methods have been used extensively as the method of analysis. Ravinder [6] stated that the reason for the widespread use of these methods is as a result of less computational time required to obtain solutions. Paulo [4], reinforced Ravinder stipulation and noted that even though the finite element methods are probably the most accurate and versatile methods, they tend to be very time consuming and require high level of knowledge not available to the common engineer..

Milosvakaya and Cherpakov [3] solved the non linear conjugate heat conduction problem using the finite difference method and showed that the scheme produces convergent results. Randall [5] proposed an unconditionally stable algorithm for numerical finite-difference solution of linear and nonlinear inverse heat conduction problems. The finite-difference analysis of ablation problems, including multidimensional problems with liquid melt layers, was facilitated by embedding them in inverse heat conduction problems. The accuracy of the finite-difference solution was assessed by comparing it with a solution using a more conventional finite-difference method and an alternative more accurate solution of the ablation problem. Hansen et al [2] solved the heat conduction problem using the finite volume method and used test examples to illustrate the effectiveness of the method as an analytical tool for temperature analysis. Schneider and Zedan [7] presented a control-volume-based formulation framework for the direct application of governing conservation principles in the determination of algebraic equations for heat conduction problems. They demonstrated convergence to the correct solution on all test problems.

It can be seen from the literature that the potential of the finite element method for obtaining solution to heat conduction problems has not been given attention. In this paper, we present the application of the continuous Galerkin finite element method to the analysis of temperature distribution in a heat conducting fiber and compare the solution obtained with that with analytical technique.

## 2.0 Governing differential equation

The governing differential equation for the temperature distribution ($T$) is given by:

$$-\frac{d}{dx}\left(K\frac{dT}{dx}\right)+\frac{2h}{r}T=\frac{2h}{r}T_m \tag{2.1}$$

$$0 < x < L$$

The associated boundary conditions are given by:

$$T(0) = T_o,\ q(L) = q \tag{2.2}$$

Where $q$ is the heat flux and is given by the expression $K\frac{dT}{dx}$.

## 2.1 Finite element modeling

To determine the temperature distribution in the fiber, we first derive the weak form of the governing differential equation given below:

$$-\frac{d}{dx}\left(K\frac{dT}{dx}\right)+\frac{2h}{r}T=\frac{2h}{r}T_m$$

$$-\frac{d}{dx}\left(K\frac{dT}{dx}\right)+\frac{2h}{r}T-\frac{2h}{r}T_m=0 \tag{2.3}$$

$$-\frac{d}{dx}\left(K\frac{dT}{dx}\right)+\frac{2h}{r}T-\frac{2h}{r}T_m=0$$

The residual for the differential equation is given as

$$R(x,a) = -\frac{d}{dx}\left(K\frac{dT}{dT}\right) + \frac{2h}{r}T - \frac{2h}{r}T_m \tag{2.4}$$

The Galerkin residual equations are given as:

$$\int_{x_a}^{x_b} R(x,a)\phi_{i(x)}dx = 0,\ i = 1,2,3,\cdots,N \tag{2.5}$$

Substituting equation (2.4) into (2.5), we obtain:

$$\int_{x_a}^{x_b}\left[-\frac{d}{dx}\left(K\frac{dT}{dx}\right) + \frac{2h}{r}T - \frac{2h}{r}T_m\right]\phi_{i(x)}dx = 0$$

$$\int_{x_a}^{x_b}\left[-\frac{d}{dx}\left(K\frac{dT}{dx}\right)\phi_{i(x)} + \frac{2h}{r}T\phi_{i(r)} - \frac{2h}{r}T_m\phi_{i(x)}\right]dx = 0 \tag{2.6}$$

Integrating the first term of equation (2.6) by parts, we obtain:

$$-\left\{\left[K\frac{dT}{dx}\phi_i\right]_{x_a}^{x_b} - \int_{x_a}^{x_b} K\frac{dT}{dx}\frac{d\phi_i}{dx}dx\right\} + \int_{x_a}^{x_b}\frac{2h}{r}T\phi_i dx - \int_{x_a}^{x_b}T_m\phi_i dx = 0$$

$$-\left[K\frac{dT}{dx}\phi_i\right]_{x_a}^{x_b} + -\int_{x_a}^{x_b}K\frac{dT}{dx}\frac{d\phi_i}{dx}dx + \int_{x_a}^{x_b}\frac{2h}{r}T\phi_i dx - \int_{x_a}^{x_b}\frac{2h}{r}T_m\phi_i dx = 0$$

$$\int_{x_a}^{x_b}K\frac{dT}{dx}\frac{d\phi_i}{dx}dx + \int_{x_a}^{x_b}\frac{2h}{r}T\phi_i dx = \int_{x_a}^{x_b}\frac{2h}{r}T_m\phi_i dx + \left[K\frac{dT}{dx}\phi_i\right]_{x_a}^{x_b} \tag{2.7}$$

Assuming an approximate solution for *T* in the form

$$T(x,a) = a_1\phi_{1(x)} + a_2\phi_{2(x)} + \ldots\ldots + a_N\phi_{N(x)} = \sum_{j=1}^{N} a_j\phi_j \tag{2.8}$$

$$\frac{dT}{dx} = \sum_{j=1}^{N} a_j\frac{d\phi_j}{dx} \tag{2.9}$$

and substituting equations (2.8) and (2.9) into (2.7) gives

$$\int_{x_a}^{x_b}K\frac{dT}{dx}\sum_{j=1}^{N}a_j\frac{d\phi_j}{dx}dx + \int_{x_a}^{x_b}\frac{2h}{r}\phi_i\sum_{j=1}^{N}a_j\phi_j = \int_{x_a}^{x_b}\frac{2h}{r}T_m\phi_i dx + \left[K\frac{dT}{dx}\phi_i\right]_{x_a}^{x_b} \tag{2.10}$$

Rearranging equation (2.10), we obtain:

$$\sum_{j=1}^{N}\int_{x_a}^{x_b}\left(K\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\right)a_j dx + \sum_{j=1}^{N}\int_{x_a}^{x_b}\left(\frac{2h}{r}\phi_i\phi_j\right)a_j dx = \int_{x_a}^{x_b}\frac{2h}{r}T_m\phi_i + \left[K\frac{dT}{dx}\phi_i\right]_{x_a}^{x_b} \tag{2.11}$$

Equation (2.11) can be written as:

$$\left[K_{ij}^e\right]\{a_j\} = \{f_i^e\} + \{\phi_i^e\} \tag{2.12}$$

Equation (2.12) is the finite element model for the problem, where

$$K_{i_j} = \int_{x_a}^{x_b}\left(K\frac{d\phi_i}{dx}\frac{d\phi_j}{dx} + \frac{2h}{r}\phi_i\phi_j\right)dx$$

$$f_i^e = \int_{x_a}^{x_b}\frac{2h}{r}T_m\phi_i dx$$

$$Q_i^e = \left[ \mathrm{K} \frac{\mathrm{dT}}{\mathrm{dx}} \phi_i \right]_{x_a}^{x_b} \tag{2.13}$$

$a_j$ = nodal degree of freedom.

The shape functions for quadratic interpolation are:

$$\phi_{1(x)}^e = \left(1 - \frac{x}{h^/}\right)\left(1 - \frac{2x}{h^/}\right)$$

$$\phi_{2(x)}^e = \frac{4x}{h^/}\left(1 - \frac{x}{h^/}\right)$$

$$\phi_{3(x)}^e = -\frac{x}{h^/}\left(1 - \frac{2x}{h^/}\right)$$

The shape functions for linear interpolation are given as

$$\phi_{1(x)}^e = \frac{x_2 - x}{x_2 - x_1}$$

$$\phi_{2(x)}^e = \frac{x - x_1}{x_2 - x_1}$$

The shape functions for both linear and quadratic interpolation can be written compactly as: $\phi_j(x) = \delta_{ji}$, where $\delta_{ji}$ is called a Kronecker delta and has the property

$$\delta_{ji} = \begin{cases} 1 & \text{if} \quad j = i \\ \\ 0 & \text{if} \quad j \neq i \end{cases}$$

## 3.0 Numerical example

Use the finite element method to find the temperature distribution in a heat conducting fiber with convection heat loss from the surface. The governing differential equation is given by:

$$-\frac{d}{dx}\left(K \frac{dT}{dx}\right) + \frac{2h}{r} T = \frac{2h}{r} T_m \qquad 0 < x < 10$$

$T(0) = 200^0 F$, $q(10) = 10BTU/sec\text{-}in^2$, $K = 2BTU/sec\text{-}in\text{-}{}^0F$, $h = 10^{-5} BTU/sec.in^2.{}^0F$, $r = 0.002in$, $T_m = 50^0 F$.

### 3.1 Solution

### 3.1.1 6- Node $C^0$ – Linear element solution

In solving the problem, we shall use a linear interpolation element for the solution. First, we will discretize the domain into mesh of two equal linear elements to five linear elements and observe the behavior of the solution. First we will need to calculate the element characteristics/stiffness matrix, noting that

$$\left[K_{ij}^e\right]\left\{a_j\right\} = \left\{\mathrm{f}_i^e\right\} + \left\{Q_i^e\right\}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left( K \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + \frac{2h}{r} \phi_i \phi_i \right) dx$$

where $x_a$ = coordinate of left end of element,

$X_b$ = coordinate of right end of element. For a choice of linear interpolation shape functions, the element characteristic matrix will be in the form

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

where $a_1$, $a_2$ = nodal degree of freedom,

$f_1$, $f_2$ = source vector terms,

$Q_1$, $Q_2$ = heat flux

$$K_{11} = \int_0^2 \left[ K \frac{d}{dx}\left(\frac{x_2 - x}{x_2 - x_1}\right)\frac{d}{dx}\left(\frac{x_2 - x}{x_2 - x_1}\right) + \frac{2h}{r}\left(\frac{x_2 - x}{x_2 - x_1}\right)\left(\frac{x_2 - x}{x_2 - x_1}\right)\right]dx$$

$$= \frac{K}{2} + \frac{2h^{/}}{r} \times \frac{2}{3} = \frac{K}{2} + \frac{4h}{3r} = \frac{2}{2} + \frac{4(10^{-5})}{3 \times 0.002} = 1.006666$$

$$K_{12} = \int_0^2 \left[ K \frac{d}{dx}\left(\frac{x_2 - x}{x_2 - x_1}\right)\frac{d}{dx}\left(\frac{x - x_1}{x_2 - x_1}\right) + \frac{2h}{r}\left(\frac{x_2 - x}{x_2 - x_1}\right)\left(\frac{x - x_1}{x_2 - x_1}\right)\right]dx$$

$$= -\frac{1}{2}K + \frac{1}{3}\left(\frac{2h}{r}\right) = -\frac{K}{2} + \frac{2h}{3r}$$

Substituting for $K$, $h$ and $r$, we obtain

$$-\frac{2}{2} + \frac{2(10^{-5})}{3 \times 0.002} = -0.996666$$

$$K_{21} = \int_0^2 \left[ K \frac{d}{dx}\left(\frac{x - x_1}{x_2 - x_1}\right)\frac{d}{dx}\left(\frac{x_2 - x}{x_2 - x_1}\right) + \frac{2h}{r}\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x_2 - x}{x_2 - x_1}\right)\right]dx$$

$$= -\frac{1}{2}K + \frac{1}{3}\left(\frac{2h}{r}\right) = -\frac{K}{2} + \frac{2h}{3r}$$

Substituting the values of $K$, $h$ and $r$ gives

$$-\frac{2}{2} + \frac{2(10^{-5})}{3 \times 0.002} = -0.996666$$

$$K_{22} = \int_0^2 \left[ K \frac{d}{dx}\left(\frac{x_2 - x}{x_2 - x_1}\right)\frac{d}{dx}\left(\frac{x_2 - x}{x_2 - x_1}\right) + \frac{2h}{r}\left(\frac{x_2 - x}{x_2 - x_1}\right)\left(\frac{x_2 - x}{x_2 - x_1}\right)\right]dx$$

$$= \frac{1}{2}K + \frac{2}{3}\left(\frac{2h}{r}\right) = \frac{K}{2} + \frac{4h}{3r}$$

Substituting for $K$, $h$ and $r$,

$$K_{22} = \frac{2}{2} + \frac{4(10^{-5})}{3 \times 0.002} = 1.006666$$

$$\mathrm{f}_i = \int_{x_a}^{x_b} \frac{2h}{r} T_m \phi_i \, dx$$

Therefore,

$$f_1 = \int_0^2 \frac{2h}{r} T_m \; \frac{x_2 - x}{x_2 - x_1} dx$$

$$= \frac{2h}{r} T_m = \frac{2\times 10^{-5}}{0.002} \times 50 = 0.5$$

$$f_2 = \int_0^2 \frac{2h}{r} T_m \; \frac{x - x_1}{x_2 - x_1} dx$$

$$= \frac{2h}{r} T_m = \frac{2\times 10^{-5}}{0.002} \times 50 = 0.5$$

In matrix form,

**Element 1**

The element characteristic matrix is given as:

$$\begin{bmatrix} 1.006666 & -0.996666 \\ -0.996666 & 1.006666 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

Since the domain is discretized into a mesh of five equal linear elements, the element characteristic matrix will be the same for all the elements. The only difference will be the associated nodal degree of freedom (primary variables i.e. nodal temperatures) and the secondary variables (heat flux). Taking advantage of this, the element characteristic matrices for the remaining elements are given below.

**Element 2**:

Element characteristic matrix is given as:

$$\begin{bmatrix} 1.006666 & -0.996666 \\ -0.996666 & 1.006666 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

where $Q_1^2$ implies the boundary flux of the first node of the second element.

**Element 3**:

Element characteristic matrix is given as:

$$\begin{bmatrix} 1.006666 & -0.996666 \\ -0.996666 & 1.006666 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} Q_1^3 \\ Q_2^3 \end{Bmatrix}$$

**Element 4:**

Element characteristic matrix is given as:

$$\begin{bmatrix} 1.006666 & -0.996666 \\ -0.996666 & 1.006666 \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} Q_1^4 \\ Q_2^4 \end{Bmatrix}$$

**Element 5**:
Element characteristic matrix is given as:

$$\begin{bmatrix} 1.006666 & -0.996666 \\ -0.996666 & 1.006666 \end{bmatrix} \begin{Bmatrix} a_5 \\ a_6 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} Q_1^5 \\ Q_2^5 \end{Bmatrix}$$

The next step is to assemble the element characteristic matrices for all the five elements. We obtain the system characteristic matrix below:

$$\begin{bmatrix} 1.00666 & -0.996666 & 0 & 0 & 0 & 0 \\ -0.996666 & 2.013332 & -0.996666 & 0 & 0 & 0 \\ 0 & -0.996666 & 2.013332 & -0.996666 & 0 & 0 \\ 0 & 0 & -0.996666 & 2.013332 & -0.996666 & 0 \\ 0 & 0 & 0 & -0.996666 & 2.013332 & -0.996666 \\ 0 & 0 & 0 & 0 & -0.996666 & 1.00666 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 10 \\ 10 \\ 10 \\ 10 \\ 5 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 + Q_1^5 \\ Q_2^5 \end{Bmatrix}$$

Imposing continuity of the flux at the interelement boundary, we have:

$$Q_2^1 + Q_1^2 = 0$$
$$Q_2^2 + Q_1^3 = 0$$
$$Q_2^3 + Q_1^4 = 0$$
$$Q_3^4 + Q_1^5 = 0$$

Also, imposing the boundary conditions we obtain

$$T(0) = 200^0 F = a_1$$
$$q(10) = Q_2^5 = 10$$

On imposition of the boundary conditions and interelement continuity of the secondary boundary variables, we obtain the following:

$$\begin{bmatrix} 1.00666 & -0.996666 & 0 & 0 & 0 & 0 \\ -0.996666 & 2.013332 & -0.996666 & 0 & 0 & 0 \\ 0 & -0.996666 & 2.013332 & -0.996666 & 0 & 0 \\ 0 & 0 & -0.996666 & 2.013332 & -0.996666 & 0 \\ 0 & 0 & 0 & -0.996666 & 2.013332 & -0.996666 \\ 0 & 0 & 0 & 0 & -0.996666 & 1.00666 \end{bmatrix} \begin{Bmatrix} 200 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{Bmatrix}$$

The above system of equations become

$$\begin{bmatrix} 2.013332 & -0.996666 & 0 & 0 & 0 \\ -0.996666 & 2.013332 & -0.996666 & 0 & 0 \\ 0 & -0.996666 & 2.013332 & -0.996666 & 0 \\ 0 & 0 & -0.996666 & 2.013332 & -0.996666 \\ 0 & 0 & 0 & -0.996666 & 1.006666 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{Bmatrix} 1 + 199.333 \\ 1.0 \\ 1.0 \\ 1.0 \\ 1.0 \\ 15.5 \end{Bmatrix}$$

$a_1 = 200.0000$
$a_2 = 196.4835$
$a_3 = 195.9067$
$a_4 = 198.2577$
$a_5 = 203.5839$
$a_6 = 211.9920$

### 3.1.2 5-Node $C^0$ – Linear element solution

For four linear elements, the assembled system matrix is shown below.

$$\begin{bmatrix} 0.808333 & -0.7958333 & 0 & 0 & 0 \\ -0.7958333 & 1.616666 & -0.7958333 & 0 & 0 \\ 0 & -0.7958333 & 1.616666 & -0.7958333 & 0 \\ 0 & 0 & -0.7958333 & 1.616666 & -0.7958333 \\ 0 & 0 & 0 & -0.7958333 & 0.808333 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0.625 \\ 1.25 \\ 1.25 \\ 1.25 \\ 1.25 \\ 0.625 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix}$$ On

applying the boundary conditions and enforcing the continuity of the secondary variables at interelement boundary, we obtain the reduced system of matrices. That is

$T(0) = a_1 = 200,$

$q(10) = Q_2^4 = 10$

$$\begin{bmatrix} 1.61666 & -0.7958333 & 0 & 0 \\ 0.7958333 & 1.616666 & -0.7958333 & 0 \\ 0 & -0.7958333 & 1.616666 & -0.7958333 \\ 0 & 0 & -0.7958333 & 10.808333 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 160.41666 \\ 1.25 \\ 1.25 \\ 10.625 \end{Bmatrix}$$

The solution is

$a_1 = 200.0000$
$a_2 = 196.0524$
$a_3 = 196.6928$
$a_4 = 201.9414$
$a_5 = 211.9630$

### 3.1.3 4-Node $C^0$ – linear Element Solution

The element characteristic matrix for the first element is given by:

$$\begin{bmatrix} 0.61111 & -0.59444 \\ -0.59444 & 0.61111 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0.8333 \\ 0.8333 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

Assembling the three element characteristic matrices, we obtain:

$$\begin{bmatrix} 0.61111 & -0.59444 & 0 & 0 \\ -0.59444 & 1.22222 & -0.59444 & 0 \\ 0 & -0.59444 & 1.22222 & -0.59444 \\ 0 & 0 & -0.59444 & 0.61111 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 0.83333 \\ 1.66666 \\ 1.66666 \\ 0.83333 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 \end{Bmatrix}$$ Imposing

the boundary conditions and interelement continuity of the secondary variables, we obtain:

$a_1 = 200.0000$
$a_2 = 195.7286$
$a_3 = 199.6247$
$a_4 = 211.9066$

### 3.1.4 3-Node $C^0$ –Linear element solution

The element characteristic matrix for the first element is given as:

$$\begin{bmatrix} 0.416666 & -0.391667 \\ -0.391667 & 0.416666 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1.25 \\ 1.25 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \end{Bmatrix}$$

Assembling the element characteristic matrices for the two matrices, we obtain the system characteristic matrix below:

$$\begin{bmatrix} 0.416666 & -0.3916667 & 0 \\ -0.391667 & 0.8333332 & -0.391667 \\ 0 & -0.3916667 & 0.416666 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 1.25 \\ 2.50 \\ 1.25 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 \end{Bmatrix}$$

Imposing the boundary conditions,

$$\begin{bmatrix} 0.8333332 & -0.3916667 \\ -0.3916667 & 0.416666 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 80.83334 \\ 11.25 \end{Bmatrix}$$

Solving the system of equations above, we get:

$a_1 = 200.0000$
$a_2 = 196.5067$
$a_3 = 211.7164$

## 3.2 Quadratic element interpolation solution

In this section, we will seek the nodal temperatures by using quadratic interpolation functions**.** Using equation (2.11) together with shape function for quadratic interpolation, we compute the quadratic element characteristic matrix. To simplify the calculation of the element characteristic matrix for different meshes, we generate an expression for the entries of the element characteristic matrix in terms of the length of each element $h^/$. Thus

$$K_{11} = \int_0^{h^/} K\left[\frac{d}{dx}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\frac{d}{dx}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\right]dx + \int_o^h \frac{2h}{r}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\left(1-\frac{x}{h}\right)\left(1-\frac{2x}{h}\right)dx$$

$$= \frac{7K}{3h^/} + \frac{2h^/}{15}\left(\frac{2h}{r}\right)$$

$$K_{12} = \int_0^{h^/} K\frac{d}{dx}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\frac{d}{dx}\left[\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\right]dx + \int_o^{h^/} \frac{2h}{r}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)dx$$

$$= -\frac{8K}{3h^/} + \frac{1}{15}h^/\left(\frac{2h}{r}\right)$$

$$K_{13} = \int_0^{h^/} K\frac{d}{dx}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\frac{d}{dx}\left[\frac{x}{h^/}\left(1-\frac{2x}{h^/}\right)\right]dx + \int_o^{h^/} \frac{2h}{r}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\frac{x}{h^/}\left(1-\frac{2x}{h^/}\right)dx$$

$$= \frac{K}{3h^/} - \frac{1}{30}h^/\left(\frac{2h}{r}\right)$$

$$K_{21}=\int_0^{h^/}\left[K\frac{d}{dx}\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\frac{d}{dx}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\right]dx+\int_o^{h^/}\left[\frac{2h}{r}\left\{\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\right\}\right]dx$$

$$=-\frac{8K}{3h^/}+\frac{1}{15}h^/\left(\frac{2h}{r}\right)$$

$$K_{22}=\int_0^{h^/}\left[K\frac{d}{dx}\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\frac{d}{dx}\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\right]dx+\int_o^{h^/}\frac{2h}{r}\left[\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\right]dx$$

$$=\frac{16K}{3h^/}+\frac{1}{15}h^/\left(\frac{2h}{r}\right)$$

$$K_{23}=\int_0^{h^/}K\frac{d}{dx}\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\frac{d}{dx}\left\{\frac{-x}{h^/}\left(1-\frac{2x}{h^/}\right)\right\}dx+\int_o^{h^/}\frac{2h}{r}\left(\frac{4x}{h^/}\left\{1-\frac{x}{h^/}\right\}\right)\left\{\frac{-x}{h^/}\left(1-\frac{2x}{h^/}\right)\right\}dx$$

$$=-\frac{8K}{3h^/}+\frac{1}{15}h^/\left(\frac{2h}{r}\right)$$

$$K_{31}=\int_0^{h^/}\left[K\frac{d}{dx}\left(\frac{-x}{h^/}\left\{1-\frac{2x}{h^/}\right\}\right)\frac{d}{dx}\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\right]dx+\int_o^{h^/}\frac{2h}{r}\left[\frac{-x}{h^/}\left(1-\frac{2x}{h^/}\right)\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)\right]dx$$

$$=\frac{K}{3h^/}-\frac{1}{30}h^/\left(\frac{2h}{r}\right)$$

$$K_{32}=\int_0^{h^/}\left[K\frac{d}{dx}\left(\frac{-x}{h^/}\left\{1-\frac{2x}{h^/}\right\}\right)\frac{d}{dx}\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\right]dx+\int_o^{h^/}\frac{2h}{r}\left[\frac{-x}{h^/}\left(1-\frac{2x}{h^/}\right)\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)\right]dx$$

$$=-\frac{8}{30h^/}+\frac{1}{15}h^/\left(\frac{2h}{r}\right)$$

$$K_{33}=\int_0^{h^/}\left[K\frac{d}{dx}\left(\frac{-x}{h^/}\left\{1-\frac{2x}{h^/}\right\}\right)\frac{d}{dx}\left(\frac{-x}{h^/}\left\{1-\frac{2x}{h^/}\right\}\right)\right]dx+\int_o^{h^/}\frac{2h}{r}\left[\frac{-x}{h^/}\left(1-\frac{2x}{h^/}\right)\left\{\frac{-x}{h^/}\left(1-\frac{2x}{h^/}\right)\right\}\right]dx$$

$$=\frac{7}{3h^/}+\frac{2}{15}h^/\left(\frac{2h}{r}\right)$$

$$\mathrm{f}_1=\int_0^{h^/}\frac{2h}{r}T_m\left(1-\frac{x}{h^/}\right)\left(1-\frac{2x}{h^/}\right)dx=\frac{2h}{r}T_m\left(\frac{1}{6}h^/\right)$$

$$\mathrm{f}_2=\int_0^{h^/}\frac{2h}{r}T_m\frac{4x}{h^/}\left(1-\frac{x}{h^/}\right)dx=\frac{2h}{r}T_m\left(\frac{2}{3}h^/\right)$$

$$\mathrm{f}_3=\int_0^{h^/}\frac{2h}{r}T_m\left(-\frac{x}{h^/}\left(1-\frac{2x}{h^/}\right)\right)dx=\frac{2h}{r}T_m\left(\frac{1}{6}h^/\right)$$

Expressing the element characteristic matrix for any element of length $h^/$ in terms of the data of the problem, we obtain the following:

$$\left[\frac{K}{3h^{/}}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{2hh^{/}}{30r}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\right]\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{2hh^{/}T_m}{6r} \\ \dfrac{4hh^{/}T_m}{3r} \\ \dfrac{2hh^{/}T_m}{6r} \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \\ Q_3^e \end{Bmatrix} \qquad (3.1)$$

### 3.2.1 3-Node $C^0$ – Quadratic element solution

Using equation (3.1) with $h^{/}$ = 10 and the data on the problem, we obtain the following system characteristic matrix:

$$\begin{bmatrix} 0.48 & -0.526666 & 0.063333 \\ -0.526666 & 1.12 & -0.526666 \\ -0.063333 & -0.526666 & 0.48 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0.8333 \\ 3.3333 \\ 0.8333 \end{Bmatrix} + \begin{Bmatrix} Q_1 \\ 0 \\ Q_2 \end{Bmatrix}$$

Imposing the boundary conditions $T(0) = a_1 = 200$ and $Q_3 = 10$, we obtain:

$$\begin{bmatrix} 1.12 & -0.52666 \\ -0.526666 & 0.48 \end{bmatrix}\begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 108.666333 \\ -1.833267 \end{Bmatrix}$$

$a_1 = 200.0000$
$a_2 = 196.7326$
$a_3 = 212.0397$

### 3.2.2 5-Node $C^0$ – Quadratic element solution

Using equation (3.1) with $h$ = 5 and the data in the problem and assembly of the resulting two element characteristic matrix, we obtain the following system characteristic matrix.

$$\begin{bmatrix} 0.94 & -1.06333 & 0.131666 & 0 & 0 \\ -1.06333 & 2.16 & -1.06333 & 0 & 0 \\ 0.131666 & -1.06333 & 1.88 & -1.06333 & 0.13166 \\ 0 & 0 & -1.06333 & 2.16 & -1.06333 \\ 0 & 0 & 0.131666 & -1.06333 & 0.94 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} = \begin{Bmatrix} 0.41666 \\ 1.666666 \\ 0.833332 \\ 1.66666 \\ 0.416666 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ 0 \\ 0 \\ 0 \\ Q_2^2 \end{Bmatrix}$$

Imposing the boundary conditions and solving the resulting matrix, we obtain

$a_1 = 200.0000$
$a_2 = 196.0830$
$a_3 = 196.7467$
$a_4 = 202.0073$
$a_5 = 212.0343$

### 3.2.3 7-Node $C^0$ – Quadratic element solution

The system characteristic matrix is given as:

$$
\begin{bmatrix}
1.4044 & -1.59777 & 0.19888 & 0 & 0 & 0 & \\
-1.59777 & 3.21777 & -1.59777 & 0 & 0 & 0 & \\
0.198888 & -1.59777 & 2.80888 & -1.59777 & 0.19888 & 0 & \\
0 & 0 & -1.59777 & 3.21777 & -1.59777 & 0 & 0 \\
0 & 0 & 0.198888 & -1.59777 & 2.80888 & -1.59777 & 0.19888 \\
0 & 0 & 0 & 0 & -1.59777 & 3.21777 & -1.59777 \\
0 & 0 & 0 & 0 & 0.19888 & -1.59777 & 1.40444
\end{bmatrix}
\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{Bmatrix}
=
\begin{Bmatrix} 0.277777 \\ 1.11111 \\ 0.55555 \\ 1.11111 \\ 0.55555 \\ 1.11111 \\ 0.277777 \end{Bmatrix}
+
\begin{Bmatrix} Q_1^1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ Q_3^3 \end{Bmatrix}
$$

$a_1 = 200.0000$
$a_2 = 196.8809$
$a_3 = 195.8047$
$a_4 = 196.7553$
$a_5 = 199.7470$
$a_6 = 204.8201$
$a_7 = 212.0466$

## 4.0 Rsesults and discussion

The temperature at the nodes for different meshes using linear and quadratic interpolation functions are shown in Table 4.1. The temperatures at points between nodes are also shown in Table 4.1. The numerical value of the calculated nodal degree of freedom shows progressive improvement of the temperature with convergence characteristic. The absolute point wise error is not greater than 7 percent for all points considered along the domain showing an admirable rate of convergence to the exact solution. Successive decrease in the length of the elements produces solutions which approach the exact solution. Figure 4.1 shows the result obtained using the linear element and the exact. It can be seen that within the ranges 2.4 – 2.6, 5.2 – 6.0 and 6.8 – 7.8, there is a marked difference between the solutions obtained using the analytical and the linear finite element solution. The reason for this deviation is as a result of the fact that the gradient of the solution within these ranges is very high. . Local mesh refinement can be used to remedy this situation if linear elements are used to discretize the domain of the fiber. Figure 4.2 shows the result obtained using quadratic finite elements compared with the exact. It can be seen that the finite element solution is admirably close to the exact at all points along the domain.

**Table 4.1**: Showing nodal temperatures along the fiber for linear, quadratic interpolation and exact solution

| **Distance/ No of Element** | **Nodal temperatures for $C^o$ – Linear Elements for different Meshes** | | | | **Nodal temperatures for $C^o$- Quadratic Element for different Meshes** | | | **Exact Nodal temperature** |
|---|---|---|---|---|---|---|---|---|
| | 5 | 4 | 3 | 2 | 1 | 2 | 3 | |
| 0 | 200 | 200 | 200 | 200 | 200 | 200 | 200 | 200 |
| 1.7 | 197.01 | 197.31 | 197.82 | 198.81 | 196.81 | 196.83 | 196.88 | 196.83 |
| 2 | 196.48 | 196.84 | 197.43 | 198.60 | 196.41 | 196.49 | 196.83 | 196.50 |
| 2.5 | 196.36 | 196.05 | 196.79 | 198.25 | 196.05 | 196.08 | 196.08 | 196.08 |
| 3.3 | 196.18 | 196.20 | 195.72 | 197.69 | 196.77 | 195.79 | 195.80 | 195.80 |
| 3.4 | 196.32 | 196.28 | 195.84 | 197.67 | 195.77 | 195.79 | 195.80 | 195.80 |
| 4 | 195.90 | 196.34 | 196.54 | 197.20 | 195.91 | 195.93 | 195.91 | 195.94 |
| 5 | 196.84 | 196.69 | 197.71 | 196.50 | 196.75 | 196.74 | 196.67 | 196.75 |
| 5.1 | 196.42 | 196.90 | 197.83 | 198.81 | 196.87 | 196.86 | 196.75 | 196.87 |
| 6 | 198.25 | 198.79 | 198.88 | 199.54 | 198.32 | 198.27 | 198.17 | 198.30 |
| 6.6 | 199.53 | 200.05 | 199.62 | 201.37 | 199.62 | 199.56 | 199.43 | 199.58 |

| Distance/ No of Element | Nodal temperatures for C° – Linear Elements for different Meshes | | | | Nodal temperatures for C°- Quadratic Element for different Meshes | | | Exact Nodal temperature |
|---|---|---|---|---|---|---|---|---|
| | 5 | 4 | 3 | 2 | 1 | 2 | 3 | |
| 6.8 | 199.10 | 200.47 | 199.93 | 201.98 | 200.12 | 200.00 | 199.74 | 200.07 |
| 7.5 | 201.45 | 201.94 | 202.94 | 204.11 | 202.07 | 202.00 | 201.61 | 202.01 |
| 8 | 203.58 | 203.94 | 204.78 | 205.63 | 203.70 | 203.63 | 203.18 | 203.63 |
| 8.5 | 205.68 | 205.95 | 206.62 | 207.15 | 205.50 | 205.44 | 204.82 | 205.44 |
| 10 | 211.992 | 211.96 | 211.71 | 211.71 | 212.03 | 212.03 | 212.04 | 212.04 |

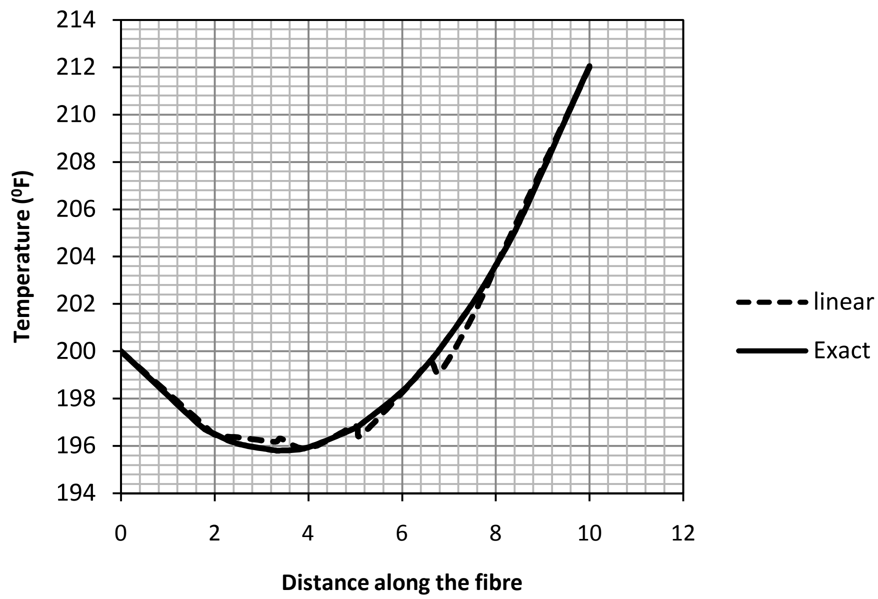


**Figure 4.1**: Graph showing temperature linear element and exact

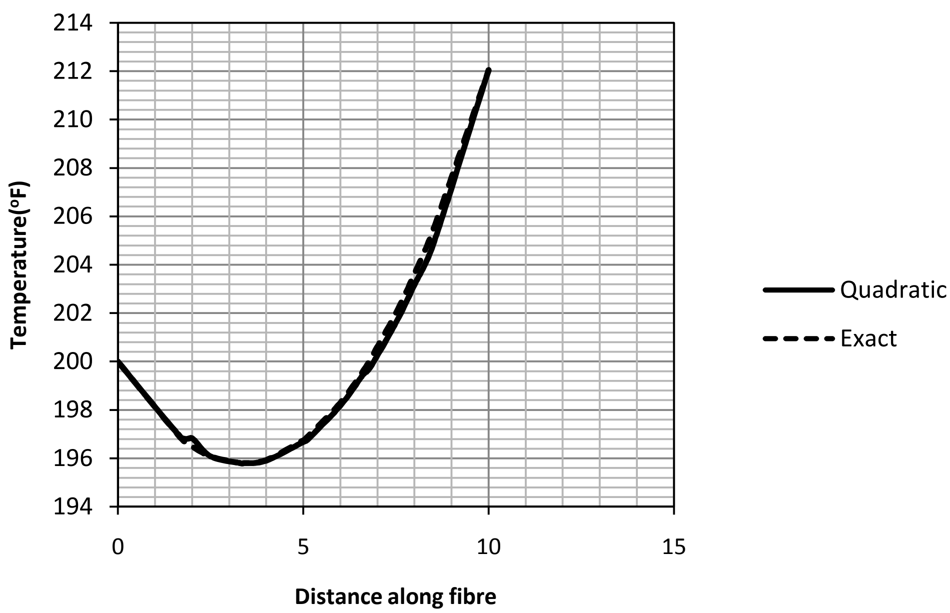


**Figure 4.2:** Graph showing temperature linear element and exact

## 5.0 Conclusion

Finite element analysis of the temperature distribution in a heat conducting fiber has been presented. It has been shown that the present method can be used to predict the temperature distribution accurately with successive mesh refinement. Three node quadratic finite elements have been shown to produce a more accurate solution to the equations governing the temperature distribution in a heat conducting fiber than linear finite elements. The potential of the finite element method has been successfully demonstrated.

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