*Journal of the Nigerian Association of Mathematical Physics*
***Volume* 14 (May, 2009), pp 299 – 304**
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# Longitudinal fields due to a rigid line inhomogeneity in a nonhomogeneous cylinder

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***Abstract***

***A non-homogeneous elastic cylinder of radius a containing a rigid line in-homogeneity under longitudinal share is analyzed for elastic compatibility. The fields were derived in a closed form with each shown to depend only on the traction prescribed on the material it represents unlike in the case of bimaterials of the same geometry under similar loading. The stress fields are not singular and satisfy conditions of continuity across the inhomogeneity thereby showing compatibility between the matrix and the inhomogeneity.***

## 1.0 Introduction

A central rigid line inhomogeneity runs through the entire length of a long nonhomogeneous elastic cylinder and covers its diameter. The cylinder has radius, a and is subjected to two prescribed longitudinal tractions $T_j$, $j$ = 1, 2 (Figure 1.1). The problem is to determine whether the inhomogeneity and the matrix are elastically compatible (see for example [1]). The solution procedure is similar to those applied in the determination of the distribution of stress in cylinders of the same geometry studied in [2, 3]. The subscripts 1 and 2 are related to materials 1 and 2 respectively.

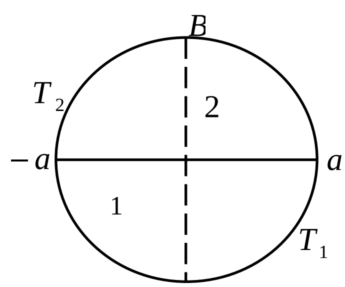


**Figure 1.1**: The Non-homogeneous material, loads and In-homogeneity

## 2.0 Solution procedure

The line inhomogeneity lies along the rays $\theta = 0, \pm\pi$. In correspondence with this mode of loading, we seek displacements $W_j(r, \theta),\ j = 1, 2$ in the *z*-direction that satisfy the following boundary value problem for the Laplace equation:

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$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)W_j(r,\theta)=0, -\pi\le\theta\le\pi \quad 0\le r\le a \tag{2.1}$$

$$W_1(r,\theta) = W_2(r,\theta) = 0 \qquad \theta=0,\pm\pi \tag{2.2}$$

$$\sigma_{jr\,z}(a,\theta) = T_j,\ (j=1,-\pi<\theta<0),\ (j=2,\ 0<\theta<\pi) \tag{2.3}$$

The problem is made suitable for analysis by method of integral transform on the upper half plane with the aid of the conformal mapping function:

$$g(z)=i\left(\frac{a+z}{a-z}\right),\ z=re^{i\theta} \tag{2.4}$$

Let $(\rho,\phi)$ be polar coordinates for the upper half plane (Figure 2.2) then

$$g(z)=u(r,\theta)+i\,v\,(r,\theta)=\rho e^{i\phi}$$

Implies $$u(r,\theta)=\frac{-2ar\sin\theta}{a^2-2ar\cos\theta+r^2},\ v(r,\theta)=\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}$$

$$\rho(r,\theta)=\left(\frac{a^2+2ar\cos\theta+r^2}{a^2-2ar\cos\theta+r^2}\right)^{\frac{1}{2}},\ \tan\phi(r,\theta)=\frac{a^2-r^2}{2ar\sin\theta} \tag{2.5}$$

Therefore, $$\frac{\partial\rho}{\partial r}(a,\theta)= 0 \text{ and } \frac{\partial\phi}{\partial r}(a,\theta)=\frac{1}{a\sin\theta} = \frac{1+\rho^2}{a\rho} \tag{2.6}$$

The stress displacement relations are:

$$\sigma_{jrz}(r,\theta)=\mu_j\frac{\partial W_j}{\partial r}(r,\theta),\ \sigma_{j\theta z}(r,\theta)=\frac{\mu_j}{r}\frac{\partial W}{\partial\theta}(r,\theta) \tag{2.7}$$

Chain rule leads to $$\frac{\partial W_j}{\partial r}(a,\theta)=\frac{\partial W_j}{\partial\phi}(\rho,\phi)\frac{\partial\phi}{\partial r}(a,\theta) \quad \theta\neq,0,\pm\pi,\quad j=1,2 \tag{2.8}$$

Using these relations we seek $W_j(\rho,\phi)$ in the problem

$$\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right)W_j(\rho,\phi)=0,\ \ 0\le\phi\le\pi,\ \rho\ge 0,\ j=1,2 \tag{2.9}$$

$$W_j\left(\rho,\tfrac{\pi}{2}\right)=0 \tag{2.10}$$

$$\frac{\partial W_1}{\partial\phi}(\rho,0)=\frac{aT_1}{\mu_1}\frac{\rho}{(1+\rho^2)} \tag{2.11a}$$

$$\frac{\partial W_2}{\partial\phi}(\rho,\pi)=\frac{aT_2}{\mu_2}\frac{\rho}{(1+\rho^2)} \tag{2.11b}$$

The behaviour of the stresses are $\sigma_{j\phi\,z}(\rho,\phi)=\sigma_{j\rho z}(\rho,\phi) = 0(1)$ as $\rho\to 0 = 0(\rho^{-2})$ as $\rho\to\infty$

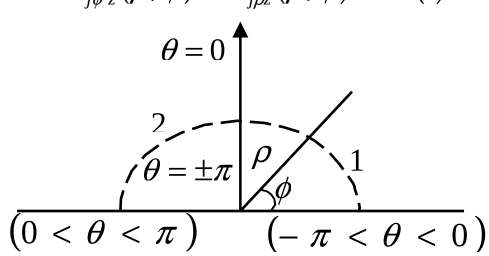


**Figure 2.2**: Corresponding segments in the upper half plane

## 3.0 Displacements from the transformed problem

When the Mellin integral transform of $W_j(\rho,\phi),\ j=1,2$ define by

$$\overline{W_j}(s,\phi)=\int_0^{\infty} W_j(\rho,\phi)\rho^{s-1}d\rho,\ -1<\operatorname{Re}\ s<1 \tag{3.1}$$

is applied to (2.10) – (3.1) the result is

$$\left(\frac{d^2}{d\phi^2}+s^2\right)\overline{W_j}(s,\phi)\ =\ 0,\ \ -1<\operatorname{Re}\ s<1 \tag{3.2}$$

$$\overline{W_j}\left(s,\tfrac{\pi}{2}\right)\ =\ 0 \tag{3.3}$$

$$\frac{\partial\overline{W_1}}{\partial\phi}(s,0)\ =\ \frac{aT_1}{\mu_1}\ H(s) \tag{3.4a}$$

$$\frac{\partial\overline{W_{21}}}{\partial\phi}(s,\pi)\ =\ \frac{aT_2}{\mu_2}\ H(s) \tag{3.4b}$$

where, by 3.2412 [4]

$$H(s)\ =\ \int_0^{\infty}\frac{\rho^s}{1+\rho}d\rho\ =\ \frac{\pi}{2\cos\frac{\pi}{2}s},\ -1<\operatorname{Re}s<1$$

Adopting a solution of (3.2) of the form

$$\overline{W_j}(s,\phi)\ =\ A_j(s)\sin s\phi+B_j(s)\cos s\phi \tag{3.5}$$

leads, through (3.3) to

$$B_j(s)\ =\ -A_j(s)\frac{\sin\frac{\pi}{2}s}{\cos\frac{\pi}{2}s} \tag{3.6}$$

Making use of (3.4a,b) and (3,6) leads to

$$A_j(s)\ =\ \frac{aT_j}{2\mu_j s}H(s)\ \text{and}\ B_j(s)\ =\frac{aT_j}{\mu_j s}\frac{\sin\frac{\pi}{2}s}{\cos\frac{\pi}{2}s}H(s)$$

which in turn yields

$$\overline{W_j}(s,\phi)\ =\ \frac{\pi\ aT_j}{2\mu_j s}\frac{\sin\left(\phi-\frac{\pi}{2}\right)s}{\cos^2{\pi/2}\,s} \tag{3.7}$$

The Mellin inversion formula gives

$$W_j(\rho,\phi)\ =\ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\overline{W_j}(s,\phi)\rho^{-s}ds,\ -1<c<1,\ j=1,2 \tag{3.8}$$

Inserting (3.7) into (3.8) yields

$$W_j(\rho,\phi)\ =\ \frac{\pi\ aT_j}{2\mu_j}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\sin\left(\phi-\frac{\pi}{2}\right)s}{s\cos^2{\pi/2}\,s}\rho^{-s}ds\right. \tag{3.9}$$

Applying residue theory and Jordan's lemma about contours in (3.9), the displacements can be written as,

$$W_j(\rho,\phi)\ =\ \frac{2aT_j}{\pi\mu_j}\left\{-\ln\rho\sum_{n=1}^{\infty}\rho^{2n-1}\frac{\sin\left(\phi-\frac{\pi}{2}\right)(2n-1)}{n2-1}-\left(\phi-\tfrac{\pi}{2}\right)\sum_{n=1}^{\infty}\rho^{2n-1}\frac{\cos\left(\phi-\frac{\pi}{2}\right)(2n-1)}{2n-1}\right.$$

$$+\sum_{n-1}^{\infty}\frac{\rho^{2n-1}}{(2n-1)^2}\sin\left(\phi-\tfrac{\pi}{2}\right)(2n-1)\},\rho<1 \qquad (3.10a)$$

$$= \frac{2aT_j}{\pi\mu_j}\left\{-\ln\rho\sum_{n=1}^{\infty}\frac{\rho^{1-2n}}{2n-1}\sin\left(\phi-\tfrac{\pi}{2}\right)(2n-1)+\left(\phi-\tfrac{\pi}{2}\right)\sum_{n=1}^{\infty}\frac{\rho^{1-2n}}{2n-1}\cos\left(\phi-\tfrac{\pi}{2}\right)(2n-1)\right.$$

$$\left.-\sum_{n=1}^{\infty}\frac{\rho^{1-2n}}{(2n-1)^2}\sin\left(\phi-\tfrac{\pi}{2}\right)(2n-1),\right\} \qquad \rho>1 \qquad (3.10b)$$

## 4.0 Conclusion

Guided by (3.10a,b) and (Figure 2.2), we observe that

$$W_1(\rho,0)=\frac{2aT_1}{\pi\mu_1}\left\{\ln\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\rho^{2n-1}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\rho^{2n-1}\right\},\rho<1 \qquad (4.1a)$$

$$=\frac{2aT_1}{\pi\mu_1}\left\{\ln\rho\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\rho^{1-2n}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\rho^{1-2n}\right\},\rho>1 \qquad (4.1b)$$

when $\phi=0,-\pi<\theta<0, r=a$ we have $W_1(\rho,0) = W_1(a,\theta)$. Now

$$\rho(a,\theta)=\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{1/2},\theta\neq 0,\pm\pi \qquad (4.2)$$

implies that, when the point *D* (see Figure 1.1) is approached from the left, the displacement from (4.1a) is

$$W_1\left(a,-\tfrac{\pi}{2}\right)=-\frac{2aT_1}{\pi\mu_1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}$$

Hence the left side of *D* is displaced in the negative *z*-direction while the right side is displaced in the positive z-direction, depicting antiplane displacement. Similarly

$$W_2(\rho,\pi) = \frac{2aT_2}{\pi\mu_2}\left\{-\ln\rho\sum_{n=1}^{\infty}\frac{(-1)}{2n-1}\rho^{2n-1}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\rho^{2n-1}\right\},\rho<1 \qquad (4.3a)$$

$$=\frac{2aT_2}{\pi\mu_2}\left\{-\ln\rho\sum_{n=1}^{\infty}\frac{(-1)}{2n-1}\rho^{1-2n}-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\rho^{1-2n}\right\},\rho>1 \qquad (4.3b)$$

On the other hand

$$W_j\left(\rho,\tfrac{\pi}{2}\right)=0 \qquad \rho<1 \text{ and } \rho>1$$

consequently

$$W_j\left(\rho,\tfrac{\pi}{2}\right)=W_j(r,\pm\pi) \quad \rho<1$$

$$= W_j(r,0) \quad \rho>1$$

implies $$W_1(r,\pm\pi)=0=W_2(r,\pm\pi) \qquad (4.4a)$$

and $$W_1(r,0)=0=W_2(r,0) \qquad (4.4b)$$

When $\phi=\pi$ (corresponding to $0<\theta<\pi,\ r=a$) we have $W_2(\rho,\pi)=W_2(a,\theta)$ and (4.2) also holds. Hence $W_2(a,\theta)=-W_1(a,\theta), \theta\neq 0,\pm\pi$ showing that displacements along the boundaries are antiplane in the z-direction.

From (3.10a,b), we derive (see, for instance [5])

$$\frac{\partial W_j}{\partial \phi}(\rho, \tfrac{\pi}{2}) = \frac{2aT_j}{\pi\mu_j}\left\{-\ln\rho\sum_{n=1}^{\infty}\rho^{2n-1}\right\} = \frac{-2aT_j}{\pi\mu_j}\left(\frac{\rho}{1-\rho^2}\right)\ln\rho, \quad \rho<1\,(\theta=\pm\pi) \quad (4.5a)$$

$$= \frac{2aT_j}{\pi\mu_j}\left\{-\ln\rho\sum_{n=1}^{\infty}\rho^{1-2n}\right\} = \frac{-2aT_j}{\pi\mu_j}\left(\frac{\rho}{\rho^2-1}\right)\ln\rho, \quad \rho>1\,(\theta=0) \quad (4.5b)$$

$$\frac{\partial W_j}{\partial \rho}(\rho, \tfrac{\pi}{2}) = 0 \qquad \rho<1 \text{ and } \rho>1 \quad (4.6)$$

From (2.5b)

$$\frac{\partial \phi}{\partial \theta}(r, \pm\pi) = \frac{-2ar}{a^2-r^2}, \qquad r<a$$

$$\frac{\partial \phi}{\partial \theta}(r, 0) = \frac{2ar}{a^2-r^2}, \qquad r<a$$

Along the inhomogeneity, the stresses in the angular direction are given by

$$\sigma_{j\theta z}(r,\theta) = \frac{\mu_j}{r}\frac{\partial W_j}{\partial \theta}(\rho, \tfrac{\pi}{2})\frac{\partial \phi}{\partial \theta}(r,\theta), \quad \theta=0,\ \pm\pi, r<a,\ j=1,2 \quad (4.7)$$

Hence

$$\sigma_{j\theta z}(r,\pm\pi) = \frac{-T_j}{\pi}\left(\frac{a}{r}\right)\ln\left(\frac{a+r}{a-r}\right), \qquad r<a \quad (4.8a)$$

$$\sigma_{j\theta z}(r,0) = \frac{-T_j}{\pi}\left(\frac{a}{r}\right)\ln\left(\frac{a+r}{a-r}\right), \quad r<a \quad (4.8b)$$

Aided by (4.6) and (4,8a,b), we deduce that the only stresses acting along the in homogeneity are nonsingular and satisfy the relations;

$$\sigma_{1\theta z}(r,\pm\pi) = \sigma_{2\theta z}(r,\pm\pi), \quad \sigma_{1\theta z}(r,0) = \sigma_{2\theta z}(r,0) \quad (4.9)$$

The conditions in (4.9) express continuity of the stresses across the inhomogeneity. Observing that (4.4a,b) asserts continuity of displacements across the in homogeneity, it follows that the matrix and the in homogeneity are elastically compatible. Therefore cracking can not set in when finite loads are applied. The fields, (3.10a,b) indicate that each material- displacement depends only on the traction applied on it and on its peculiar material constant, unlike the case of bimaterials of the same geometry with similar loading, where each material- displacement depends jointly on the tractions applied to both materials and on a material constant $\gamma = \dfrac{\mu_2-\mu_1}{\mu_2+\mu_1}$ (see, for example [2,6] ).

***References***


[1] Wang Z. Y, Zhang H.T, Chou. Y.T: Stress Singularity At the Tip of A Rigid Line Inhomogeneity Under Antiplane Shear Loading. J of Appl. Mech. Vol 53 (1986) pp 459-461

[2] Nnadi J.N: Bonded Elastic Cylindrical Solid Under Antiplane Shear .J. of Nig. Ass. Math .Phys. Vol5 (2001) pp239-250

[3] Nnadi J. N: A Neumann Problem for An Elastic Cylinder Under Out-of-Plane Loading. Global J. of Pure and Appl. Sci Vol7 (2001) pp541-544.

[4] Gradshteyn I. S and Ryzhik I.N: Tables of Integrals, Series and Products. Academic Press, New York (1965).

[5] Churchill R.V. Brown J.W. and Verhey, R.F complex Variables and Applications, Mc Graw-Hill New York (1974) pp 144 – 145.
[6] Nnadi J. N.: Elastic Bimaterial Cylinder Under Antiplane Share. J. of Nig. Ass. Mahs. Phys. Vol 14 (To Appear).