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An elastic bimaterial cylinder under anti-plane shear

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Abstract

This analysis of the stress distribution in an elastic bimaterial cylinder reveals that the stresses are nonsingular along the interface if the prescribed shear stresses remain finite. The stresses σ_{jrz} (r,θ), θ = 0, ±π, j = 1, 2 in the radial direction will varnish if the material becomes homogeneous and the loads self equilibrating. The stresses in the angular direction, $\sigma_{j\theta z}$ (r,θ), θ = 0, ±π, j = 1, 2 which tend to tear the interface are also not singular if the loads remain finite but varnish if the loads are equal.

1.0 Introduction

The deformation fields in elastic cylindrical materials have been studied in [1, 2, 3] under self equilibrated shear loads. Here, the fields are investigated for an isotropic elastic bimaterial cylinder with a perfectly bonded interface and subjected to two shear tractions T_j, j = 1, 2 (Figure 1.1). The boundary conditions in [3] lead to material constants of the type given in [4] while the continuity conditions here leads to material constants, $\gamma = \dfrac{\mu_2 - \mu_1}{\mu_2 + \mu_1}$ obtained in [5]. Fields in [3] can be shown to be special cases of the present results. The subscripts 1 and 2 refer to materials 1 and 2 respectively.

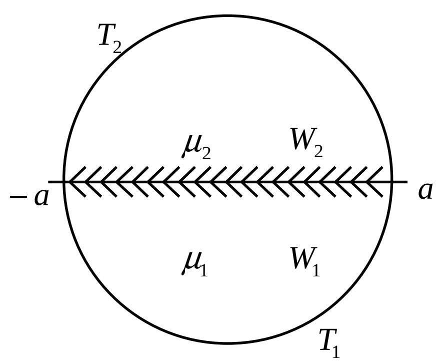

Figure 1.1: Geometry of the Problem

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The governing boundary value problem is formulated in terms of polar coordinates for the non-varnishing components of displacement $W_j(r,\theta),\ j=1,2$ as

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)W_j(r,\theta)=0,-\pi\le\theta\le\pi,\ 0\le r\le a \tag{1.1}$$

$$W_j(r,\theta)=W_2(r,\theta),\ \mu_1\frac{\partial W_1}{\partial\theta}(r,\theta)=\mu_2\frac{\partial W_2}{\partial\theta}(r,\theta)\quad \theta=0,\pm\pi \tag{1.2}$$

$$\frac{\partial W_1}{\partial r}(a,\theta)=\frac{T_1}{\mu_1},\ \frac{\partial W_2}{\partial r}(a,\theta)=\frac{T_2}{\mu_2},\ -\pi<\theta<0,0\prec\theta\prec\pi\ , \tag{1.3}$$

2.0 Solution of the boundary value problem

It is known [2] that the transformation $g(z)=\dfrac{a+z}{a-z},z=re^{i\theta}=u(r,\theta)+iv(r,\theta)$ maps the cylindrical region onto the right half plane with polar coordinates (ρ,ϕ) corresponding to $u(r,\theta)$ $=\rho\cos\phi, v(r,\theta)=\rho\sin\phi$. Therefore

$$u(r,\theta)=\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2},\ \ v(r,\theta)=\frac{2ar\sin\theta}{a^2-2ar\cos\theta+r^2}$$

So that,

$$\rho(r,\theta)=\left(\frac{a^2+2ar\cos\theta+r^2}{a^2+2ar\cos\theta+r^2}\right)^{1/2}, \tag{2.4a}$$

$$\tan\phi(r,\theta)=\frac{2ar\sin\theta}{a^2-r^2} \tag{2.4b}$$

These coordinate relationships and the transformation converts the search to that for $W_j(\rho,\phi),\quad j=1,2$ such that

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)W_j(\rho,\phi)=0-\frac{\pi}{2}\le\phi\le\frac{\pi}{2},\ \rho\ge\ 0 \tag{2.5}$$

$$W_1(\rho,\phi)=W_2(\rho,0),\ \mu_1\frac{\partial W_1}{\partial\phi}(\rho,0)=\mu_2\frac{\partial W_2}{\partial\phi}(\rho,0) \tag{2.6}$$

$$\frac{\partial W_j}{\partial\phi}\left(\rho,(-1)^j\frac{\pi}{2}\right)=(-1)^j\frac{2aT_j}{\mu_j} \tag{2.7}$$

The solution procedure requires solving the differential equation

$$\left(\frac{d^2}{d\phi^2}+s^2\right)\overline{W}(s,\phi)=0,-1<\operatorname{Re}s<1 \tag{2.8}$$

subject to the conditions

$$\overline{W_1}(s,0)\ =\ \overline{W_2}(s,0),\quad \mu_1\frac{\partial\overline{W_1}}{\partial\phi}(s,0)=\mu_2\frac{\partial W_2}{\partial\phi}(s,0) \tag{2.9}$$

$$\frac{\partial\overline{W_j}}{\partial\phi}\left(s,(-1)^j\tfrac{\pi}{2}\right)=(-1)^j\frac{2aT_j}{\mu_j}q(s)\qquad j=1,2 \tag{2.10}$$

where $\overline{W}(s,\phi)$ is the Mellin transform of $W(\rho,\phi)$ applied to (2.5) – (2.7) and defined by

$$\overline{W}(s,\phi) = \int_0^\infty W(\rho,\phi)\rho^{s-1}d\rho\ ,\ -1<\operatorname{Re}s<1$$

While by 3.2412 [6], $q(s) = \int_0^\infty \rho^s\left(1+\rho^2\right)^{-1}d\rho = \dfrac{\pi}{2\cos\frac{\pi}{2}s}, -1<\operatorname{Re}s<1$. We have used the behaviour of the stresses given by $\sigma_{\rho z}(\rho,\phi) = \sigma_{\phi z}(\rho,\phi) = 0(1)$ as $\rho\to 0$

$= 0(\rho^{-2})$ as $\rho\to\infty$

Using (2.4), (2.9),(2.10) and the solution of (2,8) in the form

$$\overline{W}_j(s,\phi) = E_j(s)\sin\phi s + F_j(s)\cos\phi s \tag{2.11}$$

leads to

$$F_1(s) = F_2(s),\ \ \mu_1\ E_1(s) = \mu_2\ E_2(s)$$

$$E_j(s) = \frac{2a\pi}{s\cos^2\frac{\pi}{2}s}\frac{1}{\mu_j}\left(\frac{T_2}{\mu_2}-\frac{T_1}{\mu_1}\right) \tag{2.12}$$

$$F_j(s) = \frac{a\pi}{s\cos\frac{\pi}{2}s}\left\{(\gamma+1)\frac{T_2}{\mu_2}-(\gamma-1)\frac{T_1}{\mu_1}\right\} \tag{2.13}$$

where γ a is characteristic constant defined by $\gamma = \dfrac{\mu_2-\mu_1}{\mu_1+\mu_2}$. The displacements are obtained from the inverse Mellin transform defined by

$$W_j(\rho,\phi) = \frac{1}{2\pi\ i}\int_{\alpha-i\infty}^{\alpha+i\infty}\overline{W}_j(s,\phi)\rho^{-s}ds\ , -1<\alpha<1,\ \ j=1,\ 2 \tag{2.14}$$

The integrand in (2.14) has $\overline{W}(s,\phi)$ deduced by substituting (2.12) and (2.13) into (2.11) therefore

$$W_j(\rho,\phi) = \pi\ a\left\{\frac{1}{\mu_j}\left(\frac{\mu_1\mu_2}{\mu_1+\mu_2}\right)\left(\frac{T_2}{\mu_2}-\frac{T_1}{\mu_1}\right)I_{(1)}(\rho,\phi)+\left[(\gamma+1)\frac{T_2}{\mu_2}-(\gamma-1)\frac{T_1}{\mu_1}\right]I_{(2)}(\rho,\phi)\right\} \tag{2.15}$$

where

$$I_{(1)}(\rho,\phi) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\sin s\phi}{s\cos^2\frac{\pi}{2}s}\rho^{-s}\ ds \tag{2.16}$$

$$I_{(2)}(\rho,\phi) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{\cos s\phi}{s\sin\pi s}\rho^{-s}\ ds \tag{2.17}$$

Application of residue theory and contour closure in accordance with Jordan's lemma for $|\phi|\le\dfrac{\pi}{2}$, leads to

$$\mathrm{I}_{(1)}(\rho,\phi) = \frac{-4}{\pi^2}\left(\ln\rho\sum_{n=1}^\infty\frac{\rho^{2n-1}}{2n-1}\sin(2n-1)\phi+\sum_{n=1}^\infty\frac{\rho^{2n-1}}{(2n-1)^2}\sin(2n-1)\phi\right.$$

$$\left.-\phi\sum_{n=1}^\infty\frac{\rho^{2n-1}}{(2n-1)}\cos(2n-1)\phi\right),\ \ \rho<1$$

$$=\frac{4}{\pi^2}\left(\ln\rho\sum_{n=1}^\infty\frac{\rho^{1-2n}}{2n-1}\sin(2n-1)\phi+\sum_{n=1}^\infty\frac{\rho^{1-2n}}{(2n-1)^2}\sin(2n-1)\phi-\phi\sum_{n=1}^\infty\frac{\rho^{1-2n}}{(2n-1)}\cos(2n-1)\phi\right),\rho>1$$

$$I_{(2)}(\rho,\phi) \quad = \quad \frac{1}{\pi}\sum_{n-1}^{\infty}\frac{(-1)^n}{n}\rho^n \cos n\phi \qquad \rho < 1$$

$$= \quad \frac{-1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\rho^{-n} \cos n\phi \qquad \rho > 1$$

3.0 Analysis of the interface field

Along the interface $\phi = 0$, $\theta = 0$ and $\theta = \pm\pi$, hence $I_{(1)}(\rho,0) = 0 \rho \geq 0$

$$I_{(2)}(\rho, 0) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\rho^n = \frac{\pi}{2}\ln\left(1-2\rho+\rho^2\right), \quad \rho < 1$$

$$= -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\rho^{-n} = -\frac{\pi}{2}\ln\left(1-2\rho^{-1}+\rho^{-2}\right), \quad \rho > 1$$

But using (2.4a), it follows that

$$1-2\rho(r,\pm\pi)+\rho^2(r,\pm\pi) = \{1-\rho(r,\pm\pi)\}^2$$

$$= \left(\frac{2r}{a+r}\right)^2, \ r < a$$

$$1-2\rho^{-1}(r,0)+\rho^{-2}(r,0) = \left(1-\rho^{-1}(r, 0)\right)^2$$

$$= \left(\frac{2r}{a+r}\right)^2, \ r < a$$

Therefore, for $r < a$, $\theta = 0, \pm\pi$, $j = 1, 2$

$$W_j(r, \theta) = 2a\left[(\gamma+1)\frac{T_2}{\mu_2}-(\gamma-1)\frac{T_1}{\mu_1}\right]\ln\left(\frac{2r}{a+r}\right) \qquad (3.1)$$

The interface stresses satisfy

$$\sigma_{j\rho z}(\rho, 0) = \mu_j\frac{\partial W_j}{\partial\rho}(\rho,0), \ \sigma_{j\phi z}(\rho,0) = \frac{\mu_j}{\rho}\frac{\partial W_j}{\partial\phi}(\rho,0), \ j = 1,2$$

For the radial stresses we note that

$$\frac{\partial\rho}{\partial r}(r,\theta) = \frac{2a}{(a-r)^2}, \frac{\partial\rho}{\partial r}(r,\pm\pi) = \frac{-2a}{(a+r)^2}$$

$$\frac{\partial I_{(2)}}{\partial\rho}(\rho,\theta) = \frac{a+r}{2\pi a}, \ r < a, \ \theta = \pm\pi \qquad (3.2a)$$

$$= \frac{1}{2\pi a}\frac{(a-r)^2}{(a+r)}, \ r < a, \ \theta = 0 \qquad (3.2b)$$

Hence from (2.15) and (3.2a,b)

$$\sigma_{jrz}(r, \theta) = \mu_j\frac{\partial W(\rho,0)}{\partial\rho}\frac{\partial\rho}{\partial r}(r,0), \ r < a,$$

$$= \mu_j\frac{\partial W_j}{\partial\rho}(\rho,0)\frac{\partial\rho}{\partial r}(r,\pm\pi), \ r < a,$$

That is

$$\sigma_{jrz}(r,\,0) = \mu_j\left[(\gamma+1)\frac{T_2}{\mu_2}-(\gamma-1)\frac{T_1}{\mu_1}\right]\frac{a}{a+r} = -\sigma_{jrz}(r,\pm\pi)$$

The exact form of the angular (opening) stresses can be found after noting that

$$\frac{\partial I_{01}}{\partial\phi}(\rho,0) = \frac{-1}{ar\pi^2}(a+r)^2\ln\left(\frac{a-r}{a+r}\right),\ r<a \quad \theta=\pm\pi$$

$$= \frac{-1}{ar\pi^2}(a-r)^2\ln\left(\frac{a+r}{a-r}\right),\ r<a \quad \theta=0$$

and that $\frac{\partial I_{(2)}}{\partial\phi}(\rho,0) = 0,\ \rho \geq 0$.

Similarly,

$$\frac{\partial\phi}{\partial\theta}(r,0) = \frac{2ar}{a^2-r^2},\ r<a\ ,$$

$$\frac{\partial\phi}{\partial\theta}(r,\pm\pi) = \frac{-2ar}{a^2-r^2},\ r<a$$

Then,

$$\sigma_{j\theta z}(r,0) = \frac{\mu_j}{r}\frac{\partial W_j}{\partial\phi}(\rho,0)\frac{\partial\phi}{\partial\theta}(r,0),\ r<a$$

$$=-\frac{2}{\pi}\frac{\mu_1\,\mu_2}{(\mu_1+\mu_2)}\left(\frac{T_2}{\mu_2}-\frac{T_1}{\mu_1}\right)\left(\frac{a}{r}\right)\left(1-\frac{r}{a}\right)^2\left(1-\frac{r^2}{a^2}\right)^{-1}\ln\left(\frac{a+r}{a-r}\right)$$

$$\sigma_{j\theta z}(r,\pm\pi) = \frac{\mu_j}{r}\frac{\partial W_j}{\partial\phi}(\rho,0)\frac{\partial\phi}{\partial\theta}(r,\pm\pi),\ r<a$$

$$= \frac{2}{\pi}\frac{\mu_1\mu_2}{(\mu_1+\mu_2)}\left(\frac{T_2}{\mu_2}-\frac{T_1}{\mu_1}\right)\left(\frac{a}{r}\right)\left(1+\frac{r}{a}\right)^2\left(1-\left(\tfrac{r}{a}\right)^2\right)^{-1}\ln\left(\frac{a-r}{a+r}\right)$$

3.0 Conclusion

The displacements have been found in closed form in terms of the bimaterial constant γ and the applied stresses. The fields along the interfaces are also obtained in a closed form and are not singular. They have a component which vanishes when the loads are self equilibrating and the material is homogenous.

Along the interface the displacements are given by (3.1). The radial stresses $\sigma_{jrz}(r,\theta)$, $\theta=0,\pm\pi,\ j=1,2$ are not singular but will varnish if the material becomes homogenous $(\mu_1=\mu_2)$ and the loads self equilibrating $(T_1=-T_2)$. The angular stresses $\sigma_{j\theta z}(r,\,\theta),\ \theta=0,\pm\pi,\ j=1,\,2$ are non-singular and varnish only when the material is homogenous and the loads are equal $(T_1=T_2)$, therefore the interface does not crack if the loads remain finite.

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