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Derivative of general Heun's equation from some properties of hypergometric functions via polynomial transformations of degrees 2,3,4,5 and 6

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Abstract

The present work determines the solutions derived from the transformation of Heun's equation to hypergeometric equation by rational substitution.

Keywords
Heun's equation, Hypergeometric functions, polynomial transformations.

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1.0 Derivative of Heun's function from some properties of hypergometric equations

Considering $H_n(a,q;\ \alpha,\beta,\gamma,\delta,\in;x)$ as the analytic solution of GHE equation [2] around $x = 0$ and normalized by $H_n(0) = 1$, we seek to answer the following questions.

(i) When is $H_n(x)$ reducible to some hypergeometric equation ${}_2F_1$?

(ii) When is $DH_n(x)$ again a $H_n(x)$ for a good choice of parameters?

Maier [4] in 2005 solved the problem (i) in full generality from the following theorem, enlarging the work of Kuiken [1]

***Theorem* 1.1** [4]

If the Heun's equation parameter values $(a,\ q\ ;\ \alpha,\beta,\gamma,\delta)$ are such that the Heun's equation is non trivial ($q \neq 0$ or $\beta \neq 0$), and all four of $t = 0, 1, a, \infty$ are singular points, then there are only seven non-composite no prefactor Heun-to-hypergeometric transformations, up to isomorphism. These seven transformations involve polynomial maps of degree, 2,3,4,3,4,5,6 respectively. A representative list gives

(1) $$H_n(2,\ \alpha\beta,\ \alpha+\beta-2\gamma+1;t) = {}_2F_1\left(\frac{\alpha}{2},\frac{\beta}{2};\gamma;1-(1-t)^2\right). \tag{1.1}$$

(2) $$H_n\left(4,\ \alpha\beta,\ \alpha,\beta,\ \frac{1}{2},\frac{2(\alpha+\beta)}{2};t\right); = {}_2F_1\left(\frac{\alpha}{2},\frac{\beta}{2};\frac{1}{2};1-(1-t)^2\left(1-\frac{t}{4}\right)\right) \tag{1.2}$$

(3) $$H_n\left(2,\ \alpha\beta,\ \alpha,\ \beta,\ \frac{(\alpha+\beta+2)}{4},\frac{\alpha+\beta}{2};t\right) = {}_2F_1\left(\frac{\alpha}{4},\frac{\beta}{4},\frac{(\alpha+\beta+2)}{4};1-4[t(2-t)-\frac{1}{2}]^2\right) \tag{1.3}$$

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(4) $$H_n\left(\frac{1}{2}+i\frac{\sqrt{3}}{2},\alpha\beta\left(\frac{1}{2}+\frac{\sqrt{3}}{6}\right);\alpha,\beta,\frac{(\alpha+\beta+1)}{3};\frac{(\alpha+\beta+1)}{3};t\right)$$

$$={}_2F_1\left(\frac{\alpha}{3},\frac{\beta}{3},\frac{\alpha+\beta+1}{3};1-\left[1-\frac{t}{\frac{1}{2}+\frac{t\sqrt{3}}{6}}\right]^3\right) \quad (1.4)$$

(5) $$H_n\left(\frac{1}{2}+i\frac{5\sqrt{2}}{4},\alpha\left(\frac{2}{3}-\alpha\right)\left(\frac{1}{2}+i\frac{\sqrt{2}}{4}\right);\alpha,\frac{2}{3}-\alpha,\frac{1}{2},\frac{1}{2};t\right)$$

$$={}_2F_1\left(\frac{\alpha}{4},\frac{2-3\alpha}{12};\frac{1}{2};1-\left(1-\frac{4t}{2+i\sqrt{2}}\right)^3\left(1-\frac{4t}{2+5i\sqrt{2}}\right)\right) \quad (1.5)$$

(6) $$H_n\left(\frac{1}{2}+i\frac{11\sqrt{15}}{90},\alpha\left(\frac{5}{6}-\alpha\right)\left(\frac{1}{2}+i\frac{\sqrt{15}}{18}\right);\alpha,\frac{5}{6}-\alpha,\frac{2}{3};\frac{2}{3};t\right)$$

$$={}_2F_1\left(\frac{\alpha}{5},\frac{5-6\alpha}{30};\frac{2}{3};-\frac{2025}{64}i\sqrt{15t}(-1+t)\left(\frac{18t-9-i\sqrt{15}}{18}\right)^3\right) \quad (1.6)$$

(7) $$H_n\left(\frac{1}{2}\pm i\frac{\sqrt{3}}{2},\alpha(1-\alpha)\left(\frac{1}{2}\pm i\frac{\sqrt{3}}{6}\right);\alpha;1-\alpha,\frac{2}{3},\frac{2}{3};t\right)$$

$$={}_2F_1\left(\frac{\alpha}{6},\frac{1}{6}-\frac{\alpha}{6};\frac{2}{3};1-4\left(\left[1-\frac{t}{\frac{1}{2}\pm\frac{i\sqrt{3}}{6}}\right]^3-\frac{1}{2}\right)^2\right) \quad (1.7)$$

2.0 Main results

Applying the derivation property of the hypergeometric functions:

$$\frac{d}{dx}\,{}_2F_1\left(a,b;c;x=R(t)=\frac{ab}{c}R'(t)\,{}_2F_1(a+1,b+1;c+1;x=R(t))\right) \quad (2.1)$$

For instance, the derivatives of the second-degree transformation *i* generates another ${}_2F_1$ with a linear prefactor.

$$\frac{d}{dx}\,{}_2F_1\left(\frac{\alpha}{2},\frac{\beta}{2};\gamma;1-(1-t^2)\right)=\frac{\beta\alpha}{2\gamma}(-1+t)\,{}_2F_1\left(\frac{\alpha+2}{2},\frac{\beta+2}{2};\gamma+1;2t-t^2\right) \quad (2.2)$$

and the pull back operator gives

(1) $$\frac{d}{dx}H_n(2,\alpha\beta;\alpha,\beta,\gamma,\alpha+\beta-2\gamma+1;t)=\frac{\beta\alpha}{2\gamma}(-1+t)$$

$$\times H_n(2,(\alpha+2)(\beta+2);\alpha+2,\beta+2,\gamma+1,\alpha+\beta-2\gamma+3;t) \quad (2.3)$$

The other six transformations work in the same way:

(2) $$\frac{d}{dx}H_n(4,\alpha\beta;\alpha,\beta,\frac{1}{2},\frac{2(\alpha+\beta)}{3};t)=\left(3-4t+t^2\right)\frac{\beta\alpha}{6}$$
$$\times H_n(4,(\alpha+3)(\beta+3);\alpha+3,\beta+3,\frac{1}{2};\frac{2}{3}(\alpha+\beta+6);t) \qquad (2.4)$$

(3) $$\frac{d}{dx}H_n(2,\alpha\beta;\alpha,\beta,\frac{\alpha+\beta+2}{4},\frac{(\alpha+\beta)}{2};t)=\frac{2\beta\alpha}{\alpha+\beta+2}(2t^2-4t+1)(-1+t)$$
$$\times H_n(2,(\alpha+4)(\beta+4);\alpha+4,\beta+4,\frac{\alpha+\beta+10}{4},\frac{(\alpha+8+\beta)}{2};t) \qquad (2.5)$$

(4) $$\frac{d}{dx}H_n\left(\frac{1}{2}+i\frac{\sqrt{3}}{2},\alpha\beta(\frac{1}{2}+\frac{\sqrt{3}}{6})\alpha,\beta,\frac{\alpha+\beta+1}{3},\frac{\alpha+\beta+1}{3};t\right)$$
$$=6\beta\alpha\left(3+i\sqrt{3}-6t\right)^2(\alpha+\beta+1)^{-1}(3+i\sqrt{3})^{-3}\times H_n\left(\frac{1}{2}+i\frac{\sqrt{3}}{2},(\beta+3)(\alpha+3)\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right);\right.$$
$$\left.\alpha+3,\beta+3,\frac{\alpha+\beta+7}{3},\frac{\alpha+\beta+7}{3};t\right) \qquad (2.6)$$

(5) $$\frac{d}{dx}H_n\left(\frac{1}{2}+i\frac{5\sqrt{2}}{4},\alpha(\frac{2}{3}-\alpha)(\frac{1}{2}+i\frac{\sqrt{2}}{4});\alpha,\frac{2}{3}-\alpha,\frac{1}{2},\frac{1}{2};t\right)$$
$$=\frac{\alpha(2-3\alpha)\left(2+i\sqrt{2}-4t\right)^2\left(2+3i\sqrt{2-4t}\right)}{3\left(2+i\sqrt{2}\right)^3\left(2+5i\sqrt{2}\right)}\times H_n\begin{pmatrix}\frac{1}{2}+i\frac{5\sqrt{2}}{4},-(\alpha+4)(\frac{10}{3}+\alpha)(\frac{1}{2}+i\frac{\sqrt{2}}{4});\\ \alpha+4,-\left(\frac{10}{3}+\alpha\right),\frac{1}{2},\frac{1}{2};t\end{pmatrix} \qquad (2.7)$$

(6) $$\frac{d}{dx}H_n\left(\frac{1}{2}+i\frac{11\sqrt{15}}{90},\alpha(\frac{5}{6}-\alpha)(\frac{1}{2}+i\frac{\sqrt{15}}{18});\alpha,\frac{5}{6}-\alpha,\frac{2}{3},\frac{2}{3};t\right)$$
$$=\frac{\sqrt{15}}{18432}i\alpha(-5+6\alpha)\left(18t-9-i\sqrt{15}\right)^2\times\left(-90t+9+i\sqrt{15}+90t^2-2it\sqrt{15}\right)$$
$$\times H_n\left(\frac{1}{2}+i\frac{11\sqrt{15}}{90},-(\alpha+5)(\frac{25}{6}+\alpha)(\frac{1}{2}+i\frac{\sqrt{15}}{18});\alpha+5,-\frac{25}{6}-\alpha,\frac{2}{3},\frac{2}{3};t\right) \qquad (2.8)$$

(7) $$\frac{d}{dx}H_n\left(\frac{1}{2}+i\frac{\sqrt{3}}{2},\alpha(1-\alpha)(\frac{1}{2}+i\frac{\sqrt{3}}{6});\alpha,1-\alpha,\frac{2}{3},\frac{2}{3};t\right)$$
$$=\alpha(1-\alpha)\frac{6\left(\left(1-\frac{t}{\frac{1}{2}+i\frac{\sqrt{3}}{6}}\right)^3-\frac{1}{2}\right)\left(1-\frac{t}{\frac{1}{2}+i\frac{\sqrt{3}}{6}}\right)^2}{3+i\sqrt{2}}\times H_n\begin{pmatrix}\frac{1}{2}+i\frac{\sqrt{3}}{2},-(\alpha+6)(\alpha+5)\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right);\\ \alpha+6,-\alpha-5,\frac{5}{3},\frac{5}{3};t\end{pmatrix} \qquad (2.9)$$

3.0 Conclusion

It is clear from the above analysis that applying the derivative operator to a Heun function transformed to hypergeometric function via polynomial transformation leads to another Heun function. This same method is being applied to Heun function via some polynomial transformation of lower degree such as t^2, $1 - t^2$, $(t - 1)^2$, $2t - t^2$, $(2t - 1)^2$, $4t(1 - t)$. This result shall be forwarded as soon as the computation is finished.

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