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Weighting factor for instantaneous source functions of a permeable interface

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Abstract

Instantaneous source functions for a layered reservoir with crossflow interface cannot be selected from already existing source functions, if the effects of the interface is to be accounted for. It is therefore necessary to modify the already existing source function. Hence, in this paper, already existing instantaneous source or Green's function is modified to account for the effect of a crossflow interface in a layered reservoir. A multiplicative weighting factor, E, is obtained which shows constant behaviour at late dimensionless flow times for a particular set of well and reservoir dimensionless parameters. Computation of dimensionless pressures using the factor shows conformity with expected behaviour for a layered reservoir with crossflow interface.

Nomenclature

$p_D = \dfrac{kh\Delta p}{141.2q\mu B}$; $t_D = \dfrac{0.001056\ kt}{\mu\phi c_t L^2}$; $i_D = \dfrac{2i}{L}\sqrt{\dfrac{k}{k_i}}$,

i = positions along x or y or z axes, *ft*;
$h_D = 1/L_D$;
Δ = drop;
p = pressure, *psi;*
k = permeability, *md;*
h = pay thickness, *ft*;
t = time, hours;
q = flow rate, STB/Day;
μ = oil viscosity, *cp*;
B = oil formation volume factor, bbl/STB;
c_t = total fluid compressibility, *1/psi*;
L = well length, *ft;*
erf = error function;
τ = dimensionless dummy time variable $t_D p'_D$ dimensionless pressure derivative.

Subscripts

x, y, z = $x, y,$ or z, directions; D = dimensionless; w = wellbore; e = external

1.0 Introduction

Instantaneous source functions are used for constructing mathematical pressure distribution models for reservoir fluid flow and reservoir simulation studies. With regions of more than one unique permeability distribution, a layered reservoir has an interface separating the different layers. Crossflow would occur across the interface if it is permeable. Writing down instantaneous source functions, to account for interfacial crossflow, requires modification

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of already derived relevant instantaneous source functions, especially those for the axes of the interface. For impermeable interfaces this modification is not necessary; they are already available in the literature [1, 2] as follows, for *z*-axis orientation of the interface:

$$s(z,t)=\frac{1}{2\sqrt{\pi\eta_z t}}e^{-(z-z_w)^2/4\eta_z t} \qquad [(1.1)$$

assuming unit production or injection rate and is valid for infinite-acting flow period. When an external boundary of any kind is felt after a long flow time, several forms of equivalent expressions to Equation 1.1 are also available [2]. For example, if the boundary along the *z*-axis is sealed at both top and bottom boundaries, then, at long flow time, Equation 1.1 becomes

$$s(z,t)=\frac{1}{h}\left[1+2\sum_{n=1}^{\infty}\exp(-\frac{n^2\pi^2 t}{h^2})\cos(\frac{n\pi z}{h})\cos(\frac{n\pi z_w}{h})\right] \qquad (1.2)$$

To adequately represent contribution from the *z*-axis, a correction to the form in Equation 1.2 is required to cater for crossflow through the interface. At the interface, the layers flow velocities and pressures are equal. The state of equilibrium is partly contributed in the dynamics in the top layer and partly from the dynamics from the bottom layer, depending on the degree of crossflow of the interface.

In this paper, a weighting factor for the z-axis is derived to account for the pro rata contribution from each layer through the interface. The factor would be valid at dimensionless times after the expiration of the radial (infinite-acting) flow period (Equation 1.1) and at the commencement of influence of flow by the interface. Figure 1.1 below is a two-layered reservoir, with a crossflow interface containing a lateral well in Layer 1. It is assumed that $k_1 > k_2$; that is, fluid (oil) in Layer 2 can be drained by the well in Layer 1, if desired.

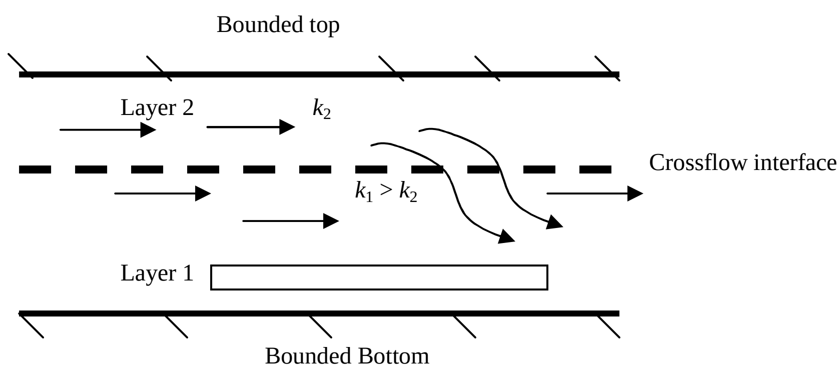

Figure 1.1: A Two-layered reservoir model with lateral wells

Fluid flow into the lateral well is contributed from three principal axes of permeability; i.e., *x*, *y*, and *z*-axes. If the well length coincides with the *x*-axis and its radius coincides with the width (along the *y*-axis), then, on the *z*-axis lies the well along the thickness of the reservoir. At very early flow times, the source function from *any* of the three axes for unit rate is given generally as [1,2]

$$s(i,t)=\frac{1}{2\sqrt{\pi\eta_i t}}e^{-(i-i_w)^2/4\eta_i t} \qquad (1.3)$$

where $i = x, y$ or *z*-direction. If the appropriate source functions have been selected, then the pressure drop anywhere in the reservoir system is given as [2].

$$\Delta p(x,y,z,\tau)=\frac{q}{\phi\mu c_t}\int_0^t s(x,\tau).s(y,\tau).s'(z,\tau)d\tau \qquad (1.4)$$

or, using dimensionless parameters

$$p_D(x_D, y_D, z_D, \tau) = 2\pi h_D \int_0^{t_D} s(x_D,\tau).s(y_D,\tau).s(z_D,\tau)d\tau \qquad (1.5)$$

The only instantaneous source function that has not been directly derived and tabulated, like the others, in the literature is $s'(z_D, t_D)$; i.e., that for the *z*-axis containing a crossflow interface.

2.0 Derivation of a weighting factor

For the fact that the interface admits fluid at the bottom of the top and releases at the top of the bottom layer, or vice versa, let the corrected instantaneous source function be

$$s'(z_D, t_D) = E_j s(z_D, t_D) \qquad (2.1)$$

where j = Layer 1 or Layer 2.

Equation 2.1 assumes a multiplicative correction for the existing $s(z_D, t_D)$. As mentioned before, the nature of $s'(z_D, t_D)$ depends on the boundary-type being modeled. The factor E_j will be derived for each layer dimensionless pressure distribution for complete duplication of the interface effect. Therefore, equation 1.5 is now best written as

$$p_{Dj}(x_D, y_D, z_D, \tau) = 2\pi h_D E_j \int_0^{t_D} s(x_D,\tau).s(y_D,\tau).s(z_D,\tau)d\tau \qquad (2.2)$$

But the dimensionless wellbore pressure in each layer is unity as flow continues. That is

$$p_{wDj} = \frac{p_j - p(x_D y_{D,} z_D, t_D)}{p_j - p(x_{wD} y_{wD,} z_{wD}, t_D)} = 1 \qquad (2.3a)$$

since the reservoir pressure drop is equal to the layer's wellbore pressure drop. The general expression for Equation 2.2 for constant terminal pressure case is now therefore written as

$$2\pi h_D E_j \int_0^{t_D} s(x_{wD},\tau).s(y_{wD},\tau).s'(z_{wD},\tau)d\tau = 1 \qquad (2.3b)$$

Assuming that the LHS of Equation 2.3b is expandable [3, 4] like other space variables, then equation 2.3b can now be represented as

$$2\pi \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} h_{D\,j} E_j \int_0^{t_D} s(x_D,\tau).s(y_{wD},\tau).s'(z_{wD},\tau)d\tau = 1. \qquad (2.4)$$

where the *E*'s are the weighting factors. By virtue of the orthogonality [4] of the source functions, and considering Layers j = 1 and 2,

$$E_j = \frac{(I_1 + I_2)}{2\pi(II_1 + II_2)} \qquad (2.5)$$

where

$$I_1 = h_{D1} \int_{x_{wD}}^{x_{eD}} \int_0^{y_{eD}} \int_0^{h_{D1}} \left[\int_0^{t_D} s_1(x_D,\tau).s_1(y_D,\tau).s_1(z_D,\tau)d\tau \right] dz_D dy_D dx_D \qquad (2.5a)$$

$$I_2 = h_{D2} \int_{x_{wD}}^{x_{eD}} \int_0^{y_{eD}} \int_{h_{D2}}^{h_D} \left[\int_0^{t_D} s_2(x_D,\tau).s_2(y_D,\tau).s_2(z_D,\tau)d\tau \right] dz_D dy_D dx_D \qquad (2.5b)$$

$$II_1 = h_{D1}^2 \int_{x_{wD}}^{x_{eD}} \int_0^{y_{eD}} \int_0^{h_{D1}} \left[\int_0^{t_D} s_1(x_D,\tau).s_1(y_D,\tau).s_1(z_D,\tau)d\tau \right]^2 dz_D dy_D dx_D \qquad (2.5c)$$

$$II_2 = h_{D2}^2 \int_{x_{wD}}^{x_{eD}} \int_0^{y_{eD}} \int_{h_{D2}}^{h_D} \left[\int_0^{t_D} s_2(x_D,\tau).s_2(y_D,\tau).s_2(z_D,\tau)d\tau \right]^2 dz_D dy_D dx_D \quad (2.5d)$$

E_j are therefore the desired weighting factor for computing the actual source function contribution from the z-axis for the crossflow system.

2.1 Description of Source Functions in Eqs. 9 and 10

Note that in Equations 2.5a to 2.5d, all the external boundaries are assumed to have been in effect. However, if the well is centrally located along the *x*-axis and the width is infinitesimally small compared with the width of the reservoir (as is in actual practice), then the integration of all the Green's functions simply yield the equivalent source functions, with just positions along the axes substituted [1, 2]. That is,

$$\int_{x_{wD}}^{x_{eD}} s_j(x_D,t_D)dx_D = s_j(x_D,t_D)dx_D \quad (2.6)$$

and

$$\int_0^{y_{eD}} s_j(y_D,t_D)dy_D = s_j(y_D,t_D)dy_D \quad (2.7)$$

The most commonly used forms of equations 2.6 and 2.7 are

$$s(x_D,t_D) = \frac{1}{4}\left[erf\left(\frac{\sqrt{k/k_x} + x_D}{2\sqrt{t_D}} \right) + erf\left(\frac{\sqrt{k/k_x} - x_D}{2\sqrt{t_D}} \right) \right] \quad (2.8)$$

for central well location on the x-axis, and

$$s(y_D,t_D) = \sqrt{\frac{k}{k_y}} \frac{e^{-(y_D - y_{wD})^2/4t_D}}{\sqrt{\pi t_D}} \quad (2.9)$$

for an anisotropic reservoir during the early linear flow period. At long dimensionless flow times, when all the external lateral boundaries and the interface are felt, then for sealing lateral boundaries,

$$s(x_D,t_D) = \frac{1}{x_{eD}}\left[1 + \frac{2x_{eD}}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \exp(-\frac{n^2\pi^2 t_D}{x_{eD}^2}) \sin\frac{n\pi}{x_{eD}} \cos\frac{n\pi x_{wD}}{x_{eD}} \cos\frac{n\pi x_D}{x_{eD}}) \right] \quad (2.10)$$

and

$$s(y_D,t_D) = \frac{1}{y_{eD}}\left[1 + 2\sum_{m=1}^{\infty} \exp(-\frac{m^2\pi^2 t_D}{y_{eD}^2}) \cos\frac{m\pi y_{wD}}{y_{eD}} \cos\frac{m\pi y_D}{y_{eD}}) \right] \quad (2.11)$$

The source functions for the *z*-axis are however different. Under the condition of crossflow, the bottom layer is modeled as a reservoir with a recharge through the interface (top boundary) while the bottom boundary is impermeable to flow. The source function is therefore an infinite plane source in an infinite slab reservoir with a partially recharging (constant-pressure, permeable) upper boundary written as

$$s(z_D,t_D) = \frac{1}{h_{D2}} \sum_{n=1}^{\infty} \exp(-\frac{(2n+1)^2\pi^2\tau}{4h_{D2}^2}) \cos\frac{(2n+1)\pi z_{D2}}{h_{D2}} \cos\frac{(2n+1)\pi z_{wD2}}{h_{D2}} \quad (2.12)$$

The top layer is modeled as a reservoir with a constant-pressure (recharge) bottom boundary, similar to a bottom water. The source function is therefore an infinite plane source in an infinite slab reservoir with a partially recharging (constant-pressure, permeable) bottom boundary written as

$$s_1(z_D,t_D) = \frac{1}{h_{D1}} \sum_{n=1}^{\infty} \exp(-\frac{(2n-1)^2\pi^2 t_D}{4h_{D1}^2}) \sin\frac{(2n-1)\pi z_{wD1}}{2h_{D1}} \sin\frac{(2n-1)\pi z_{D1}}{2h_{D1}} \quad (2.13)$$

With all the source functions specified, Equation 2.1 is now complete with the factor *E* taken as a constant parameter outside the integral. The theorem of superposition in time is used to evaluate the integral whereas the factor *E* is computed as a constant parameter for a constant dimensionless time. All integrations can be performed numerically according to [5 and 6].

The range of validity of E in pressure distribution computation, as given by equation 2.1, is given with respect to dimensionless flow time as $t_{De} \le t_D \le t_{Dzf}$, where t_{De} is the dimensionless time to end of infinite-acting flow period and t_{Dzf} is the dimensionless time to end of the interface effect. [7, 8 and 9] give expressions for estimating t_{De} and can be modified to estimate t_{Dzf}. The magnitude of E depends strictly on both wellbore and interface properties. If the interface is reasonably long, permeable and homogeneous, it would produce the effect of a fully recharging boundary. On the other hand, an interface with low permeability and short length would produce the effect of an incompletely penetrated or incompletely completed well, if pressure transients were monitored from one of the layers. To understand the effects of crossflow on E, let

$$t_{D1} = \beta t_{D2} \qquad (2.14)$$

where β is a correction factor which accounts for difference in response times of the layers to the same transient regime. For instance, if individual layer dimensionless pressure distribution were to be measured from the same transient history, and the layers have different reservoir, wellbore and fluid properties, then, if dimensionless time Layer 2 would respond to the transient is t_{D2}, Layer 1 would be responding to the same transient at dimensionless time βt_{D2}. The same response character would also be responsible for the difference in amount of fluid that can be produced from each layer for the same dimensionless flow time. If β = 1, it means that the two layers probably have exactly the same properties and therefore the same response character. The importance of this factor is most appreciable in fluid recovery project design and predictions. If dimensional parameters are used, then from Equation 2.14,

$$\beta = (\phi_1 c_{t1} \mu_1 L_1^2 k_2)/((\phi_2 c_{t2} \mu_2 L_2^2 k_1) \qquad (2.15)$$

according to the definitions of the dimensionless parameters. The factor is either estimated from Equation 2.14 or from type cure matching. The summations appearing in Equation 2.5 are obtained for n = l = m = 1, i.e., at the wellbore, where Equation 2.3 is valid.

3.0 Results and Discussion

A few sets of wellbore parameters are now assumed below to compute the values of E for an isotropic reservoir and β = 1. The sources, $s(x_D, t_D)$, $s_1(z_D, t_D)$, $s_2(z_D, t_D)$ and $s(y_D, t_D)$ from Equations 2.8, 2.9, 2.10 and 2.11 respectively, were entered into Equation 2.5, i.e.; laterally infinite reservoir case. Results are shown below in Tables 3.1 and 3.2 and Figure 3.1.

Table 3.1: Effects of dimensionless width on E

$h_{D1} = h_{D2} = 1.0$; $x_{D1} = x_{D2} = 0.732$, $z_{D1} = 0.7$, $z_{D2} = 0.05$

t_D	$y_{wD1} = 5 \times 10^{-5}$ $y_{WD2} = 5 \times 10^{-5}$	10^{-4} 5×10^{-5}	10^{-3}	10^{-4} 10^{-1}	10^{-2} 10^{-3}	10^{-3} 10^{-4}	10^{-3} 10^{-1}
10^{-2}	22.62	22.62	22.62	211.76	22.28	22.62	211.73
10^{-1}	6.52	6.52	6.52	9.19	6.51	6.52	9.19
10^{-0}	2.48	2.48	2.48	2.57	2.48	2.48	2.57
10^{1}	1.20	1.20	1.20	1.21	1.20	1.20	1.21
10^{2}	1.02	1.02	1.02	1.02	1.02	1.02	1.02
10^{3}	0.996	0.996	0.996	0.996	0.996	0.996	0.996
10^{4}	0.994	0.994	0.994	0.994	0.994	0.994	0.994

Table 3.2: Effects of layers thickness on E

$y_{wD1} = y_{D2} = 10^{-5}$; $z_{wD1} = z_{wD2} = h_D/2, h_{D1}/2, h_{D2} = h_{D2} = 1.0$

t_D	h_{D1}= 0.2 h_{D2} = 0.2	0.4 0.4	0.6 0.6	0.8 0.8	1.0 1.0
10^{-2}	262.42	-272.7	74.89	29.44	0.00
10^{-1}	69.33	616.90	20.05	8.38	0.00
10^{-0}	29.94	118.25	7.33	3.18	0.00
10^{1}	16.15	65.91	3.60	1.55	0.00
10^{2}	13.91	57.27	3.03	1.31	0.00
10^{3}	13.68	56.40	3.00	1.29	0.00

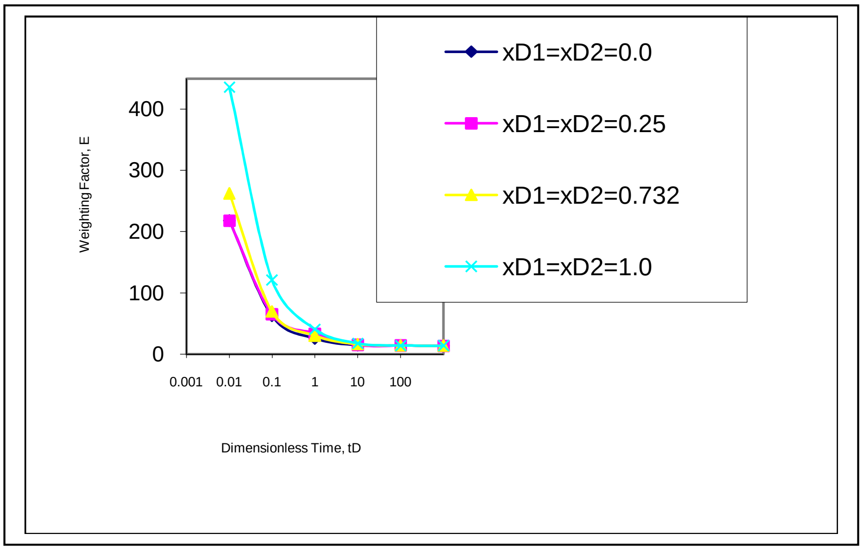

Figure 3.1: Effects of flow point on E

It is observed from Tables 3.1 and 3.2 that at very early dimensionless times, E varies widely but inversely with dimensionless time. This behaviour is followed by gradual decreases until final stabilization at constant values for all sets of dimensionless well widths considered. Since E is not valid at early dimensionless times, it therefore means that at the expiration of infinite-acting flow, E remains fairly constant for all dimensionless flow times. Layers thickness produces sinusoidal effects on E while well widths do not affect E considerably. Furthermore, the results shown in Figure 3.1 that, no matter where the well is perforated (x_D = 0.0, 0.25, 0.732 or 1.0), the same effects are observed on E for the same well stand-off and width at late dimensionless times.

Dimensionless pressures computed using Equation 2.2 are tabulated in Table 3.3 for a laterally infinite two-layered reservoir. In the computation, $h_{D1} = h_{D2} = h_D$ to ensure that the entire reservoir is exposed to flow. Well 1 is the monitoring well. But in computing the values of the modification factor, the dimensionless pay thickness of each layer was used. Thus, as an extended reservoir, the parameters in the integrals in Equation 2.2 are total pay thickness, total permeabilities and $t_{D1} = \beta t_{D2}$, where $\beta = 1.0$. Computed dimensionless pressures show that a slope of $2\Delta p_D/\Delta ln t_D \approx 1.151$ comparable to the results of a single layer reservoir case in [8], especially just at expiration of early transients. At late dimensionless times, however, there is a reduction in the values of p_D, because the effects of layering is now being felt.

The dimensionless pressure is now less than those of an equivalent single layer reservoir as revealed in [9] for crossflow layered reservoirs with horizontal wells and [10] for vertical wells with crossflow layers. Thus, there can be more rapid economic depletion for the equivalent crossflow reservoir than for an equivalent single layer system without crossflow.

Table 3.1: Computed dimensionless pressures using modification factor, *E*

Dimensionless Times, t_D	**Dimensionless Pressures, p_D**			
	Case 1 $h_D = 0.8$ $h_{D1} = 0.4$ $z_{wD1} = 0.2$ $z_{D2} = 0.2$ $z_{D1} = 0.2$ $z_{wD2} = 0.2$	**Case 2** $h_D = 1.2$ $h_{D1} = 0.6$ $z_{wD1} = 0.3$ $z_{D2} = 0.3$ $z_{D1} = 0.3$ $z_{wD2} = 0.3$	**Case 3** $h_D = 1.0$ $h_{D1} = 0.5$ $z_{wD1} = 0.25$ $z_{D2} = 0.25$ $z_{D1} = 0.25$ $z_{wD2} = 0.25$	**Case 4** $h_D = 3.0$ $h_{D1} = 1.5$ $z_{wD1} = 0.75$ $z_{D2} = 0.75$ $z_{D1} = 0.75$ $z_{wD2} = 0.75$
0.000001	1.3601	2.0402	1.70015	5.1004
0.00001	1.8206	2.7310	2.2758	6.8274
0.0001	2.2812	3.4217	2.8515	8.5543
0.001	2.7417	4.1125	3.4271	10.2813
0.01	3.2022	4.8033	4.0028	12.0082
0.1	3.6627	5.4941	4.5784	13.7351
1	4.1232	6.1848	5.1541	15.4621
10	4.5837	6.8756	5.7297	17.1890
100	5.0443	7.5664	6.3053	18.9160
1000	5.5048	8.2572	6.8610	20.6249
10000	5.9653	8.9479	7.4566	22.3698

4.0 Conclusion

When selecting instantaneous source or Green’s functions for a crossflow-layered reservoir with horizontal well, a modified form of the functions for the axis containing the crossflow interface has been derived. The influence of both wellbore and layers’ properties was investigated. Instantaneous source and Green’s functions for both early and late dimensionless flow times were used to test the behaviour of the derived modification factor, *E*, for every set of wellbore and layers’ properties. Results obtained show that

(1) the modification factor is invalid for early dimensionless times (infinite-acting) flow period as revealed in computed dimensionless pressures for equivalent set of wellbore and layers’ properties.

(2) The modification factor attains a constant value from inception of late dimensionless flow time.

(3) As a multiplicative modification factor, only relevant source and /or Green’s functions are simply substituted into Eq.10 to duplicate the reservoir external boundaries of interest.

(4) The modification factor is unaffected by perforation location and well width.

References

[1] Carslaw, H.S. and Jaeger, J.C.: *Conduction of Heat through Solids*, 2nd Ed. Oxford University Press, London, England, 1959.

[2] Gringarten,A.C. and Ramey,H.J.Jr. (1973): “The Use of Source and Green’s Functions in Solving Unsteady-Flow Problems in Reservoirs,” SPE *Trans*.; AIME, 255-285.

[3] Greenberg, M.D.: *Application of Green’s Functions in Science and Engineering*, Prentice Hall, Inc. Englewood Cliff, 1971, p. 45-46**.**

[4] Katz, L.M. and Tek, M.R.: “A Theoretical Study of Pressure Distribution and Fluid Flux in Bounded Stratified Porous System with Crossflow,” *Trans*.; AIME (1962)225, P. 68-82.

[5] Adewole, E.S.: “The Use of Gauss-Legendre and Gauss-Laguerre Quadratures in Solving Problems involving Oil Flow in Horizontal Wells, *Abacus: J. of the Mathematical Ass. of Nigeria*, vol. 34, Number 2B, Mathematical Series, Sept. 2007, p. 380-386

[6] Carnahan, B., Luther, H.A., and Wilkes, J.O.: *Applied Numerical Methods*, John Wiley and Sons Publishers, New York, 1969.

[7] Odeh, S.A. and Babu, D.K.: "Transient Flow Behaviour of Horizontal Wells: Pressure Drawdown and Buildup Analysis," *SPEFE* (March 1990) 7 –15, Trans., AIME, 289.

[8] Ozkan, E. and Raghavan, R.: "Performance of Horizontal Wells Subject to Bottom Water Drive," *SPERE* (Aug. 1990) 375-383.

[9] Adewole, E.S. (2004): "The Use of Pressure Functions to Model Pressure Distribution in a Layered Reservoir with Lateral Wells, Ph.D. Thesis, Univ. of Benin, Nigeria.

[10] Russell, D.G. and Prat, M.: "Performance of Layered Reservoirs with Crossflow-Single-Compressible-Fluid Case" *SPEJ*, March 1962, p. 53 – 67.