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A new derivation of continuous collocation multistep methods using power series as basis function

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Abstract

Some derivations of Continuous Linear Multistep Methods are given in this paper. The paper provides the use of both collocation and interpolation techniques to obtain the schemes. Rather than using Chebyshev polynomials as basis function as it was always done in the past, we introduced the use of direct form of power series as an alternative to the derivation of these schemes. Multistep Methods have over the years been one of the most popular and acceptable methods for generating solutions to initial value problems of Ordinary Differential Equations.

Keywords
Collocation, Multistep, Continuous schemes

1.0 Introduction

Over the years, several authors have considered the collocation methods as ways of generating numerical solutions to Ordinary Differential Equations (ODEs). The collocation method is dated as far back as 1956 in the work carried out by Lanczos [1] (see also Herman, [2]. Lanczos introduced the standard collocation method with some selected points. However, Fox and Parker introduced the use of Chebyshev polynomials in the collocating the existing method which was captioned as the Lanczos-Tau method Fox and Parker, [3]. Also, Ortiz [4] went on to discuss the general Tau method which was later extended by Onumanyi and Ortiz [5] to a method known as the Collocation-Tau method. The Standard Collocation method with method of selected points provides a direct extension of the Tau method to linear ODEs with non polynomial coefficients. The Collocation -Tau method however uses the Chebyshev perturbation terms to select the collocation points. Okunuga and Onumanyi [6, 7] gave the generalized Tau method which permits exact fractional values in the computation with more than one τ- term as perturbation on the right hand side of the linear differential equation. This was later extended to Non-linear differential equations with some linearization being introduced on the Tau method by Okunuga and Sofoluwe [8].

Other researchers such as Onumanyi et al. [9], Adeniyi and Alabi [10], Fatokun [11] have however introduced some other variants of the collocation methods which recently led to some continuous collocation approach. The introduction of the continuous collocation schemes as against the discrete schemes includes the fact that better global error can be estimated and approximations can be equally obtained at all interior points. Furthermore, the

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introduction of the continuous collocation method has been able to bridge the gap between the discrete collocation methods and the conventional multistep methods. Thus it is possible to write the Linear Multistep Methods (LMMs) in form of some continuous collocation schemes.

Various techniques have been suggested for the derivation of LMMs. In this paper we propose the use of generalized power series as a basis function on the collocation method which will lead to some continuous collocation schemes and are easily linked to the LMMs.

2.0 General collocation method

It is a known fact that the Linear Multistep Methods have over the years being very useful in generating solutions to IVP in ODEs. Consider the Initial Value problem (IVP)

$$y'(x) = f(x, y(x)) \qquad y(x_0) = y_0 \tag{2.1}$$

The LMMs for solving the IVP (2.1) can be developed by some collocation and interpolation techniques. The general *k*-step method or LMM of step number *k* given by Lambert [12] is written as,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j}, \qquad \alpha_k \neq 0. \tag{2.2}$$

where α_j and β_j are uniquely determined and h is the step length, such that, $x_{k+n} - x_k = nh$

The LMMs generate discrete multistep schemes (2.2) which are used for solving the IVP (2.1). There have been various forms of the LMM which were discussed by Henrici (1962) [13], Lambert [12], Fatunla [14], as well as Butcher [15].

Many of the Linear Multistep Schemes given by (2.2) have been proved to have satisfied some stability conditions. Due to the nature of various problems, other variants of the LMM do exist in the literature. Some of these include the hybrid LMM, second derivative LMM and the generalized Multiderivative LMM Okunuga, Lambert, and Butcher, [18; 17; and 10]. These are variously developed to improve the accuracy of the results being obtained when solving the IVP (2.1) and other higher order linear ODEs. Thus, the Continuous Collocation approach, which require collocating at some points x_k of the Linear k-step method (2.2) is rewritten in continuous form as,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\sum_{j=0}^{k} \beta_j (x) f_{n+j} \tag{2.2a}$$

where $\beta_j(x)$ is now defined as a function of *x* and it is continuously differentiable at least once. In this paper, we developed some continuous multistep collocation methods with some collocation points taken at the grid points using some form of Series or polynomials $(x - x_j)^k$ as the basis function.

2.1 Power series collocation

The Taylor polynomial is the ultimate in osculation. For a single argument x_o, the values of the polynomial and its first *n* derivatives are required to match those of a given function *y*(*x*). That is $p^{(r)}(x_0) = y^{(r)}(x_0),\ r = 0,1,2,\cdots,n$. The existence and uniqueness of such a polynomial is well known and they are classical results of analysis. The Taylor's formula settles the existence issue directly by exhibiting such polynomial in the form

$$p(x) = \sum_{r=0}^{n} \frac{y^{(r)}(x_0)}{r!}(x - x_0)^r \tag{2.3}$$

Analytic function has the property that for $n \to \infty$, the approximate function p(x) reduces to y(x).

Based on this argument, we propose a polynomial series in form of (2.3) as the basis function for deriving the LMM.

In the work done to date, Onumanyi et al. [9] derived some finite difference methods that lead to some LMMs for the solution of IVPs in ODEs of the form (2.1). By appropriate selection of points for both interpolation and collocation, many important classes of finite difference methods were recovered and new methods were generated. These authors also used a collocating function of the form

$$y(x) = \sum_{k=0}^{M} \alpha_k x^k$$

Adeniyi and Alabi [10] derived the continuous LMMs by using some Chebyshev polynomial functions as basis functions. These authors proposed a collocating function of the form

$$Y(x) = \sum_{j=0}^{M} a_j T_j \left(\frac{x - x_k}{h} \right), \; x_k \le x \le x_{k+p}$$

where $T_j(\xi)$ are some Chebyshev functions which are used as basis function in their work. We however propose in this paper a basis function of the form

$$y(x) = \sum_{r=0}^{M} a_j (x - x_k)^r \tag{2.4}$$

which is in form of (2.3) and will be shown to have identical results and methods with the work of other previous authors. We are able to generate more methods by our new approach which makes this different from other previous works and simpler than that of Adeniyi and Alabi [10]. The series (2.4) proposed here shall be used for both collocation and interpolation techniques that such methods may require.

The use of this basis function will permit us to derive some continuous LMM of various orders and consequently the discrete formulas are also obtained. We shall make comparison of our methods with those generated by using the Chebyshev polynomials. The power series (2.4) permits smooth functions in which a_j's are suitably determined by collocation techniques, so as to generate some LMMs in continuous form.

3.0 Linear multistep methods

The LMM have over the years been very useful in generating solutions to IVP in ODE. There has been various form of the LMM which were derived by Lambert [12, 17], Fatunla [14], as well as Butcher [15]. All of these schemes given in the form of equation (2.2) are in discrete form. Among the existing methods of deriving the LMM in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of the LMM. These various techniques were developed over the years because no single approach can really produce all existing multistep schemes. There are still some schemes that can be written in the form of (2.2) by fixing certain values for the coefficients of y_{n+j} and f_{n+j} which may not be easily obtained by the techniques listed above. Hence the need to seek more approaches of deriving these all important schemes.

It is also useful to note that many of these schemes have been proved to have satisfied some stability conditions. Due to the nature of various problems, other variants of the LMM exist also in literature. Some of these include the second derivative LMMs. These are equally developed to improve the accuracy of the numerical results being obtained when solving the IVP.

In this paper we shall develop the continuous form of the LMMs, which permits collocating at various points rather than the usual discrete formulas.

The derivation given in this paper is quite different from the usual techniques given by Lambert and Butcher, but will end up to yield the same LMMs which in this paper could be written both in discrete and continuous forms.

Definition* 3.1*:

Consider the IVP

$$y'(x) = f(x, y(x)), \, y(x_0) = y_0, \; x \in [x_0, y_0], \; y(x), \; f(x, y) \in R^m$$

Where we assume that there exist some Lipschitz constant L such that,

$$\| f(x, y) - f(x, z) \| \le L \| y - z \|, \; \forall (x, y), \, (x, z) \in (x_0, y_0) \times R^m$$

This implies that the IVP has a unique solution.

***Definition* 3.2**

The first characteristics polynomial of the LMM (2.2) is given by

$$\rho(\xi) = \sum_{r=0}^{\infty} \alpha_r \, \xi^r$$

The methods for which $\rho(\xi) = \xi^k - \xi^{k-1}$ are called Adams methods, while those that have $\rho(\xi) = \xi^k - \xi^{k-2}$ are the Nystrom methods.

3.1 Derivation of continuous LMM

Consider the polynomial function

$$Y(x) = \sum_{j=0}^{M} a_j (x - x_k)^j \cong y(x) \qquad x_k \le x \le x_{k+p} \tag{3.1}$$

over each of the sub-interval $[x_k, x_{k+p}]$ of $[a,b]$ where M is appropriately chosen and $Y(x)$ approximates the solution y(x) of equation (2.1). This shall be used as basis function to derive some LMMs in the continuous form.

The technique which is being employed for the derivation of the schemes is by setting $M = n+1$ in equation (3.1). Hence we write

$$Y(x) = \sum_{j=0}^{n+1} a_j (x - x_k)^j \cong y(x) \qquad x_k \le x \le x_{k+p} \tag{3.2}$$

as the trial or basis function. This satisfies the unperturbed ODE,

$$Y'(x) = f(x, y(x)), \quad Y(x_k) = Y_k \qquad x_k \le x \le x_{k+p} \tag{3.3}$$

We shall collocate equation (3.3) at $(n+1)$ points x_{k+j}, $j = 0,1,2,...,n$ and also interpolate the trial polynomial (3.2) at x_k to give the required $(n+2)$ equations for the unique determination of a_j. Doing this, we write

$$\left.\begin{array}{l} f(x_{k+j}) = f_{k+j}, \; j = 0,1,2,.... \\ Y(x_k) = Y_k \end{array}\right\} \tag{3.4}$$

To derive a one step LMM, we set $n = 1$, in (3.2), so that

$$Y(x) = a_0 + a_1(x - x_k) + a_2(x - x_k)^2 \tag{3.5}$$

From (3.4), we have

$$Y'(x_k) = f_k, \quad Y'(x_{k+1}) = f_{k+1} \tag{3.6}$$
$$Y(x_k) = Y_k$$

Using (3.5) in (3.6), we obtain the three equations

$$Y(x_k) = a_0 = Y_k$$
$$Y'(x_k) = a_1 = f_k$$
$$Y'(x_{k+1}) = a_1 + 2a_2(x_{k+1} - x_k) = f_{k+1}$$

Representing this in the matrix form to get,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & h & 0 \\ 0 & h & 2h^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} Y_k \\ hf_k \\ hf_{k+1} \end{pmatrix}$$

On solving, since a_o and a_1 are known, then a_2 is determined as

$$a_2 = \frac{f_{k+1} - f_k}{2(x_{k+1} - x_k)}$$

Substituting in (3.5), we obtain

$$Y(x)=Y_k + f_k(x-x_k)+\frac{f_{k+1}-f_k}{2(x_{k+1}-x_k)}(x-x_k)^2 \qquad (3.7)$$

Equation (3,7) is the continuous formulation of a one-step method. To obtain its discrete form, we evaluate at $x=x_{k+1}$ to obtain

$$Y_{k+1}-Y_k=\frac{h}{2}(f_{k+1}+f_k) \qquad (3.8)$$

Equation (3.8) is the well-known trapezoidal method of order 2 and it is an implicit one-step scheme. On the other hand if we put $x=x_{k+2}$ in (3.7), we shall equally obtain the explicit 2-step scheme as:

$$Y_{k+2}-Y_k=2hf_{k+1} \qquad (3.9)$$

This is a two –step explicit LMM and it is called the Mid-point Rule. We can also derive some other two step methods by setting $n=2$ in (3.2). That is

$$Y(x)=\sum_{j=0}^{3}a_j(x-x_k)^j \qquad (3.10)$$

By collocating and interpolating at x_k and x_{k+1}, we have

$$Y'(x_k)=f_k$$
$$Y'(x_{k+1})=f_{k+1}$$
$$Y'(x_{k+2})=f_{k+2}$$
$$Y(x_k)=Y_k$$

This in turn is rewritten to include a_i as,

$$Y(x_k)=a_0=Y_k$$
$$Y'(x_k)=a_1=f_k$$

$$Y'(x_{k+1})=3a_3(x_{k+1}-x_k)^2+2a_2(x_{k+1}-x_k)+a_1$$
$$Y'(x_{k+2})=3a_3(x_{k+2}-x_k)^2+2a_2(x_{k+2}-x_k)+a_1$$

Representation on a matrix yields the system:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & h & 0 & 0\\ 0 & h & 2h^2 & 3h^3\\ 0 & h & 4h^2 & 12h^3\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}Y_k\\ hf_k\\ hf_{k+1}\\ hf_{k+2}\end{pmatrix}$$

Solving for a_2 and a_3, since a_o and a_1 are already known above, then we obtain

$$a_2=\frac{1}{2(x_{k+1}-x_k)}\left[-\tfrac{1}{2}f_{k+2}+2f_{k+1}-\tfrac{3}{2}f_k\right]$$

$$a_3=\frac{1}{6(x_{k+1}-x_k)^2}(f_{k+2}-2f_{k+1}+f_k)$$

Substituting for a_0, a_1, a_2, a_3 in equation (3.10), we obtain

$$Y(x)=Y_k+f_k(x-x_k)+\frac{1}{2h}\left[-\frac{1}{2}f_{k+2}+2f_{k+1}-\frac{3}{2}f_k\right](x-x_k)^2 + \frac{1}{6h^2}[f_{k+2}-2f_{k+1}+f_k](x-x_k)^3 \qquad (3.11)$$

Evaluating at $x=x_{k+2}$, we obtain the discrete form of equation (3.11) after simplification as

$$Y_{k+2}-Y_k=\frac{1}{3}h[f_{k+2}+4f_{k+1}+f_k] \qquad (3.12)$$

which is the Simpson's method. Equation (3.11) is the continuous formulation of the discrete scheme (3.12) and it is known to be of order 4, Henrici [13].

3.2 The *N*-step optimal order

We shall at this point consider in a general form a LMM of optimal order with n steps. We consider our trial polynomial (3.2), that is, $Y(x)=\sum_{j=0}^{n+1} a_j (x-x_k)^j$. On substituting into the IVP (2.1) and collocating at $n+1$ points x_{k+j}, $j=0,1,2,\ldots,n$ and interpolating at x_k to give a $(n+2)$ systems of equations for the unique determination of $a_j's$, $j=0,1,2,\ldots,n+1$, we shall obtain,

$$Y(x)=a_{n+1}(x-x_k)^{n+1}+a_n(x-x_k)^n+a_{n-1}(x-x_k)^{n-1}+\ldots$$
$$+a_3(x-x_k)^3+a_2(x-x_k)^2+a_1(x-x_k)+a_0$$
$$Y'(x)=(n+1)a_{n+1}(x-x_k)^n+na_n(x-x_k)^{n-1}+(n-1)a_{n-1}(x-x_k)^{n-2}+\ldots$$
$$+3a_3(x-x_k)^2+2a_2(x-x_k)+a_1=f(x,y)$$

Interpolating at x_k and Collocating at $x_k,\ x_{k+1},\ x_{k+2},\ldots,\ x_{k+n}$, we obtain

$Y(x_k)=a_0=Y_k$

$Y'(x_k)=a_1=f_k$

$$\left.\begin{aligned}Y'(x_{k+1})&=(n+1)a_{n+1}(x_{k+1}-x_k)^n+na_n(x_{k+1}-x_k)^{n-1}+\ldots\\&\quad+3a_3(x_{k+1}-x_k)^2+2a_2(x_{k+1}-x_k)+a_1\end{aligned}\right\}=f_{k+1}$$
$$\left.\begin{aligned}Y'(x_{k+2})&=(n+1)a_{n+1}(x_{k+2}-x_k)^n+na_n(x_{k+2}-x_k)^{n-1}+\ldots\\&\quad+3a_3(x_{k+2}-x_k)^2+2a_2(x_{k+2}-x_k)+a_1\end{aligned}\right\}=f_{k+2}$$
$$\left.\begin{aligned}Y'(x_{k+3})&=(n+1)a_{n+1}(x_{k+3}-x_k)^n+na_n(x_{k+3}-x_k)^{n-1}+\ldots\\&\quad+3a_3(x_{k+3}-x_k)^2+2a_2(x_{k+3}-x_k)+a_1\end{aligned}\right\}=f_{k+3}$$
$$\left.\begin{aligned}Y'(x_{k+n})&=(n+1)a_{n+1}(x_{k+n}-x_k)^n+na_n(x_{k+n}-x_k)^{n-1}+\ldots\\&\quad+3a_3(x_{k+n}-x_k)^2+2a_2(x_{k+n}-x_k)+a_1\end{aligned}\right\}=f_{k+n+1}$$

This leads to $a_0=Y_k$, $ha_1=hf_k$

$(n+1)a_{n+1}h^{n+1}+na_nh^n+\ldots+3a_3h^3+2a_2h^2+a_1h=hf_{k+1}$

$2^n(n+1)a_{n+1}h^{n+1}+2^{n-1}na_nh^n+\ldots+2^2.3a_3h^3+2.2a_2h^2+a_1(h)=hf_{k+2}$

$3^n\cdot(n+1)a_{n+1}h^{n+1}+3^{n-1}\cdot na_nh^n+\ldots+3^2\cdot 3a_3h^3+3\cdot 2a_2h^2+a_1h=hf_{k+3}$

$(n+1)\cdot n^n\cdot a_{n+1}h^{n+1}+n^n\cdot a_nh^n+\ldots+n^2\cdot 3a_3h^3+n\cdot 2a_2h^2+a_1h=hf_{k+n+1}$

Representing this in a matrix form, the following is deduced.

$$\begin{pmatrix}1&0&0&0&0&0&0\\0&h&0&0&0&0&0\\0&h&2h^2&3h^3&\cdots&nh^n&(n+1)h^{n+1}\\0&h&2\cdot 2h^2&2^2.3h^3&\cdots&2^{n-1}\cdot nh^n&2^{n\cdot}\cdot(n+1)h^{n+1}\\0&h&3\cdot 2h^2&3^2.3h^3&\cdots&3^{n-1}.nh^n&3^n\cdot(n+1)h^{n+1}\\0&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&h&n\cdot 2h^2&n^2.3h^3&\cdots&n^{n-1}.nh^n&.(n+1)\cdot n^nh^{n+1}\end{pmatrix}\begin{pmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\ \\a_{n+1}\end{pmatrix}=\begin{pmatrix}Y_k\\hf_k\\hf_{k+1}\\hf_{k+2}\\hf_{k+3}\\\vdots\\hf_{k+n+1}\end{pmatrix}$$

This can be solved using Numerical solvers for the independent solutions of $a_0,\ a_1,\ldots,\ a_{n+1}$. When the values of $a_0,\ a_1,\ldots,\ a_{n+1}$ are substituted into the basis function, the result obtained gives the continuous formulation of the Linear multistep method, while specific evaluation at point x_k gives the discrete equivalent.

3.3 Derivation of classes of Adams methods

We shall further construct some continuous schemes which yield classes of Adams Methods. The Adams methods are broadly classified into two, namely Adam-Bashforth (explicit) schemes and Adam-Moulton (implicit) schemes. Thus for the IVP (2.1), the technique involves is by setting $M = n$ in (3.1) so that the trial or basis function becomes

$$Y(x) = \sum_{j=0}^{n} a_j (x - x_k)^j \cong y(x),\ x_k \le x \le x_{k+p} \tag{3.13}$$

This satisfies the unperturbed equation (3.3). Collocating (3.3) at n points x_{k+j}, $j = 0,1,2,...,(n-1)$ and interpolating the trial polynomial (3.13) at the x_{k+n-1} to give the required $(n+1)$ equations for the unique determination of a_j, $j = 0,1,2,...,n$. To derive a one step Adam-Bashforth scheme, we set $n = 1$ in (3.13) and using equation (3.3), we have

$$Y'(x_k) = f_k$$
$$Y(x_k) = Y_k$$

Using the basis function (3.13), we equally obtain

$$Y'(x_k) = a_1 = f_k,\ Y(x_k) = a_0 = Y_k \tag{3.14}$$

On substituting a_0 and a_1 in the basis function (3.13), we have the continuous formulation as

$$Y(x) = Y_k + f_k (x - x_k) \tag{3.15}$$

Evaluating at x_{k+1}, we obtain the Euler explicit method

$$Y_{k+1} = Y_k + hf_k \tag{3.16}$$

Similarly for a 2-step Adam-Bashforth method, set $n = 2$ in (3.13). Interpolating (3.13) at $x = x_{k+1}$ and collocating the derivative of (3.13) at $x = x_k$ *and* x_{k+1}, we obtain the following equations.

$$Y(x_{k+1}) = a_0 + a_1(x_{k+1} - x_k) + a_2(x_{k+1} - x_k)^2$$
$$Y'(x) = a_1 + 2 \cdot a_2 (x - x_k)$$
$$Y'(x_k) = a_1 = f_k$$
$$Y'(x_{k+1}) = a_1 + 2 \cdot a_2 (x_{k+1} - x_k)$$

Solving we get

$$a_0 = Y_{k+1} - \frac{h}{2}[f_{k+1} + f_k]$$

$$a_2 = \frac{1}{2h}(f_{k+1} - f_k)$$

This gives the continuous method as:

$$Y(x) = Y_{k+1} - \frac{h}{2}[f_{k+1} + f_k] + f_k(x - x_k) + \frac{1}{2h}[f_{k+1} - f_k](x - x_k)^2 \tag{3.17}$$

Evaluating at $x = x_{k+2}$, we obtain the discrete form as:

$$Y_{k+2} = Y_{k+1} + \frac{h}{2}(3f_{k+1} - f_k) \tag{3.18}$$

Equation (3.17) is the continuous formulation of the two-step Adams Bashforth scheme and the discrete form is given by equation (3.18).

3.4 *N*-Step Adams-Bashforth scheme.

Using the trial polynomial (17) and substituting into the ordinary differential equation (1), collocating at n points x_{k+j}, $j = 0,1,2,...,n-1$ and interpolating at x_{k+n-1} to get an $(n+1)$ systems of equations for the unique determination of a_j's, $j = 0,1,2,...,n$.

Doing this, we obtain,

$$Y(x) = a_n(x-x_k)^n + a_{n-1}(x-x_k)^{n-1} + \ldots + a_3(x-x_k)^3 + a_2(x-x_k)^2 + a_1(x-x_k) + a_0$$

$$Y'(x) = na_n(x-x_k)^{n-1} + (n-1)a_{n-1}(x-x_k)^{n-2} + \ldots + 3a_3(x-x_k)^2 + 2a_2(x-x_k) + a_1 = f(x,y).$$

Interpolating at x_{k+n-1} and Collocating at $x_k,\ x_{k+1},\ x_{k+2},\ \ldots,\ x_{k+n}$, we obtain

Interpolation:

$$Y(x_{n+k-1}) = Y_{n+k-1} = a_n(x_{k+n-1}-x_k)^n + a_{n-1}(x_{k+n-1}-x_k)^{n-1} + \ldots$$
$$+ a_3(x_{k+n-1}-x_k)^3 + a_2(x_{k+n-1}-x_k)^2 + a_1(x_{k+n-1}-x_k) + a_0$$

Collocation:

$$Y'(x_k) = a_1 = f_k$$
$$Y'(x_{k+1}) = na_n(x_{k+1}-x_k)^{n-1} + \ldots + 3a_3(x_{k+1}-x_k)^2 + 2a_2(x_{k+1}-x_k) + a_1 = f_{k+1}$$
$$Y'(x_{k+2}) = na_n(x_{k+2}-x_k)^{n-1} + \ldots + 3a_3(x_{k+2}-x_k)^2 + 2a_2(x_{k+2}-x_k) + a_1 = f_{k+2}$$
$$Y'(x_{k+3}) = na_n(x_{k+3}-x_k)^{n-1} + \ldots + 3a_3(x_{k+3}-x_k)^2 + 2a_2(x_{k+3}-x_k) + a_1 = f_{k+3}$$
$$Y'(x_{k+n-1}) = na_n(x_{k+n-1}-x_k)^{n-1} + \ldots + 3a_3(x_{k+n-1}-x_k)^2 + 2a_2(x_{k+n-1}-x_k) + a_1 = f_{k+n-1}$$
$$(n-1)^n h^n a_n + (n-1)^{n-1}h^{n-1}a_{n-1} + \ldots + (n-1)^3h^3a_3 + (n-1)^2h^2a_2 + (n-1)ha_1 + a_0 = Y_{k+n-1}$$

Multiplying the collocations by h, we obtain

$$ha_1 = hf_k$$
$$na_nh^n + \ldots + 3a_3h^3 + 2a_2h^2 + a_1h = hf_{k+1}$$
$$2^{n-1}na_nh^n + \ldots + 2^2.3a_3h^3 + 2.2a_2h^2 + a_1(h) = hf_{k+2}$$
$$3^{n-1}\cdot na_nh^n + \ldots + 3^2\cdot 3a_3h^3 + 3\cdot 2a_2h^2 + a_1h = hf_{k+3}$$
$$n(n-1)^{n-1}a_nh^n + \ldots + (n-1)^2\cdot 3a_3h^3 + (n-1)\cdot 2a_2h^2 + a_1h = hf_{k+n-1}$$

Representing this on a matrix, the following is derived,

$$\begin{pmatrix} 1 & (n-1)h & (n-1)^2h^2 & (n-1)^3h^3 & \cdots & (n-1)^nh^n \\ 0 & h & 0 & 0 & \cdots & 0 \\ 0 & h & 2h^2 & 3h^3 & \cdots & nh^n \\ 0 & h & 2\cdot 2h^2 & 2^2\cdot 3h^3 & \cdots & 2^{n-1}\cdot nh^n \\ 0 & h & 3\cdot 2h^2 & 3^2\cdot 3h^3 & \cdots & 3^{n-1}\cdot nh^n \\ & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & h & (n-1)\cdot 2h^2 & (n-1)^2\cdot 3h^3 & \cdots & .n(n-1)^{n-1}h^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \\ a_n \end{pmatrix} = \begin{pmatrix} Y_{k+n-1} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \\ \vdots \\ hf_{k+n-1} \end{pmatrix}$$

This can similarly be solved using some numerical solvers for the independent solutions of $a_0, a_1, \ldots, a_n$. When the values of $a_0, a_1, \ldots, a_n$ are substituted in the basis function, the result obtained is called the continuous formulation of the Linear multistep method for its discrete equivalent.

3.5 Adams-Moulton schemes

In a similar manner we can formulate the classes of Adams-Moulton schemes in continuous form by using our series as the basis function. The Adams-Moulton schemes are LMMS that are implicit in nature and are often used as the corrector to the Adams-Bashforth schemes. As such the derivation of this is equally important. Since the Adams-Moulton methods are implicit, the appropriate technique is to use the trial or basis function of the form.

$$Y(x) = \sum_{j=0}^{n+1} a_j(x-x_k)^j \cong y(x),\ x_k \le x \le x_{k+p} \tag{3.19}$$

If we set $n = 2$, we can obtain a 2-step implicit method as follows:

$$Y'(x_k) = f_k$$
$$Y'(x_{k+1}) = f_{k+1}$$
$$Y'(x_{k+2}) = f_{k+2}$$
$$Y(x_{k+1}) = Y_{k+1}$$

Using the basis function (23) for n = 2, we get,

$$\left.\begin{aligned} Y(x_{k+1}) &= a_0 + ha_1 + h^2a_2 + h^3a_3 = Y_{k+1} \\ Y'(x_k) &= a_1 = f_k \\ Y'(x_{k+1}) &= a_1 + 2ha_2 + 3h^2a_3 = f_{k+1} \\ Y'(x_{k+2}) &= a_1 + 4ha_2 + 12h^2a_3 = f_{k+2} \end{aligned}\right\} \tag{3.20}$$

Solving (3.20), we obtain

$$a_3 = -\frac{1}{3h^2}\left[f_{k+1} - \tfrac{1}{2}f_k - \tfrac{1}{2}f_{k+2}\right]$$
$$a_2 = \frac{1}{4h}\left[4f_{k+1} - 3f_k - f_{k+2}\right]$$
$$a_0 = Y_{k+1} - \frac{h}{12}\left[5f_k + 8f_{k+1} - f_{k+2}\right]$$

Substituting for a_0, a_1, a_2 and a_3 in (3.19) with $n = 2$, we obtain

$$Y(x) = Y_{k+1} - \frac{h}{12}\left[5f_k + 8f_{k+1} - f_{k+2}\right] + f_k(x - x_k) + \frac{1}{4h}\left[4f_{k+1} - 3f_k - f_{k+2}\right](x - x_k)^2 - \frac{1}{3h^2}\left[f_{k+1} - \frac{f_k}{2} - \frac{f_{k+2}}{2}\right](x - x_k)^3 \tag{3.21}$$

Evaluating at $x = x_{k+2}$ and simplify we obtain

$$Y_{k+2} = Y_{k+1} + \frac{h}{12}\left[5f_{k+2} + 8f_{k+1} - f_k\right] \tag{3.22}$$

Equation (3.21) is the continuous formulation of Adams-Moulton scheme, while equation (3.22) is its discrete formula which is of order 3. Several other methods can be obtained by the same techniques demonstrated above.

4.0 Numerical example

Consider the system of ODE,

$$\frac{d}{dx}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -0.5y \\ 4 - 0.3z - 0.1y \end{pmatrix}, \begin{pmatrix} y(0) = 4 \\ z(0) = 6 \end{pmatrix} \tag{4.1}$$

The differential equation is known to have an analytical solution

$$y = 4e^{-0.5x} \text{ and, } z = \frac{-4e^{-\frac{x}{2}} + 40}{3} - 6$$

Using the Numerical Schemes, we obtain, $Y(x) \cong y(x)$ and $Z(x) \cong z(x)$. The approximate solution of the System of Ordinary Differential Equation (4.1) is solved using the following schemes derived above

- 2-Step Optimal Order Scheme (PTD2)
- 2-Step Adams-Bashforth Scheme (ABS2)
- 2-Step Adams-Moulton Scheme (AMM2)

The numerical results obtained by each Scheme with their errors are presented on Tables 4.1 – 4.4 below.

Table 4.1: Table for the Numerical Solution of $Y\ (x)$

X	**PTD2 (*y*)**	**ABS2 (*y*)**	**AMM2 (*y*)**	**Exact *y(x)***
0.1	3.804918	3.804918	3.804918	3.804918
0.2	3.619354	3.619355	3.619355	3.619355
0.3	3.442822	3.442823	3.442831	3.442832
0.4	3.274916	3.274917	3.274923	3.274924
0.5	3.115197	3.115198	3.115202	3.115203
0.6	2.963268	2.963269	2.963272	2.963273
0.7	2.818748	2.81875	2.818752	2.818752
0.8	2.681278	2.681279	2.681281	2.681281
0.9	2.550511	2.550511	2.550513	2.550513
1	2.426121	2.426121	2.426123	2.426123

Table 4.2: Table for the Error of Numerical Solution of $Y\,(x)$

	Error of y_i		
X	**PTD2 (*y*)**	**ABS2 (*y*)**	**AMM2 (*y*)**
0.1	6.14E-07	5.61E-07	3.71E-08
0.2	3.73E-07	3.37E-07	2.34E-08
0.3	9.4E-06	8.89E-06	9.19E-07
0.4	7.92E-06	6.79E-06	8.87E-07
0.5	5.78E-06	4.86E-06	7.49E-07
0.6	4.41E-06	3.74E-06	6.04E-07
0.7	4.05E-06	2.74E-06	4.93E-07
0.8	3.24E-06	2.32E-06	4.12E-07
0.9	2.59E-06	2.23E-06	3.73E-07
1	2.24E-06	1.92E-06	2.19E-07

Table 4.3:- Table for the Numerical Solution of $z(x)$

X	**PTD2 (*z*)**	**ABS2 (*z*)**	**AMM2 (*z*)**	**Exact *z(x)***
0.1	6.064983	6.064964	6.06502	6.065027
0.2	6.126846	6.126829	6.126876	6.126883
0.3	6.185693	6.185679	6.185715	6.185722
0.4	6.241674	6.241653	6.241686	6.241692
0.5	6.293974	6.294903	6.294926	6.294932
0.6	6.344732	6.345551	6.345571	6.345576
0.7	6.393009	6.393732	6.393746	6.393749
0.8	6.438913	6.438607	6.439545	6.439573
0.9	6.482669	6.482369	6.48314	6.483162
1	6.524402	6.524004	6.524607	6.524626

Table 4.4: Table for the Error of Numerical Solution of $z(x)$

	Error of z_i		
X	**PTD2 (*z*)**	**ABS2 (*z*)**	**AMM2 (*z*)**
0.1	4.40E-05	6.34E-05	7.34E-06
0.2	3.73E-05	5.37E-05	7.05E-06
0.3	2.94E-05	4.29E-05	6.84E-06
0.4	1.79E-05	3.92E-05	6.24E-06
0.5	9.58E-04	2.88E-05	5.52E-06
0.6	8.44E-04	2.48E-05	4.76E-06
0.7	7.41E-04	1.74E-05	3.32E-06
0.8	6.60E-04	9.66E-04	2.78E-05
0.9	4.93E-04	7.93E-04	2.17E-05
1	2.24E-04	6.22E-04	1.89E-05

5.0 Conclusion

It has been shown that continuous collocation methods for solving IVPs can equally be derived through the approach discussed in this paper. It is not compulsory to use the special function as a basis function to derive these schemes. A simple power series used in this paper is sufficed for such derivations. It should be noted that the Optimal Order produces a better result than the Adams-Bashforth Schemes of the same step but the Adams-Moulton scheme is most accurate. The schemes generated are stable and consistent.

The results generated in this paper could be compared with the continuous collocation schemes generated by other authors cited in this work. All our derivations agreed with known discrete formulas.

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