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An accurate scheme by block method for third order ordinary differential equations.

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Abstract

A block linear multistep method for solving special third order initial value problems of ordinary differential equations is presented in this paper. The approach of collocation approximation is adopted in the derivation of the scheme and then the scheme is applied as simultaneous integrator to special third order initial value problem of ordinary differential equations. This implementation strategy is more accurate and efficient than those given when the same scheme is applied over overlapping intervals in predictor-corrector mode. Furthermore, the new block method possesses the desirable feature of Runge-Kutta method of being self-starting and eliminates the use of predictor- corrector method. Experimental results confirm the superiority of the new scheme over the existing methods.

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Keywords

Linear multistep methods (LMMs); P-stability; Zero-stability; Third order; IVPs; Odes; Interval of periodicity; Predictor-corrector

1.0 Introduction

The special third order initial value problems (IVPs) given as

$$y''' = f(x, y,) \;\; \mathrm{y(a)} = \mathrm{y}_0, \;\; \mathrm{y}'(\mathrm{a}) = \eta_0, \;\; \mathrm{y}''(\mathrm{a}) = \eta_1 \qquad (1.1)$$

is considered. We assume that the numerical solution is required on a given set of mesh

$$\pi = \{x_n / x_n = a + nh, h = x_{n+1} - x_n, n = 0,1,\cdots, N\} \text{ where } N = b\text{-}a/h.$$

Recent research works in this area includes Awoyemi and Idowu [1] who studied a class of hybrid collocation method for general third order of ordinary differential equations. Awoyemi [2] developed a *p*-stable linear multistep method for general third order initial value problems of ordinary differential equations but the implementation strategy is in predictor-corrector mode. Like other linear multistep methods and other standard methods, are usually applied to the initial value problems as a single formula but the drawbacks of the methods are well known. Firstly they are not self-starting; secondly, they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution model Yusuph [3]. Moreover, the overlapping creates a disadvantage because the numerical model fails to represent the solution uniquely elsewhere than the mesh-points. For boundary value problems, this is an important criticism of the linear multistep methods in favour of the finite element methods (see Jennings [4]).

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Jator [5] presented a class of initial value methods for the direct solution of second order initial value problems. In his work, LMMs with continuous coefficients were obtained and applied as simultaneous numerical integrators to $y'' = f(x, y, y')$. The implementation strategy is more efficient than those given in Awoyemi [6] which are applied over overlapping intervals in predictor-corrector mode. Vigo-Aguiar and Ramos [7] constructed a variable step-size implementation of multistep Methods for, $y'' = f(x, y, y')$. Yusuph and Onumanyi [8] derived new multiple Finite Difference Methods (FDMs) through multistep collocation for $y'' = f(x, y)$. Fatokun and Onumanyi [9] also derived second and fourth order two-step discrete finite difference methods by collocation for the first approximation and combined them with the Numerov method for a direct application to general second order initial value problem of ODEs.

This paper therefore proposes 5-step block scheme for the solution of special third order ordinary differential equations which eliminates the use of predictors by providing sufficiently accurate simultaneous difference equations from a single continuous formula and its derivative.

2.0 Methodology

A power series of a single variable *x* in the form

$$P(x) = \sum_{j=0}^{\infty} a_j x^j \tag{2.1}$$

is used as the basis or trial function, to produce the approximate solution as

$$y(x) = \sum_{j=0}^{k+2} a_j x^j \tag{2.2}$$

$$a_j \in R,\ j = 0(1)k + 2,\ y \in C^m(a, b) \subset P(x).$$

Assuming an approximate solution to (1.1) in the form of (2.2) whose high derivatives are

$$y'(x) = \sum_{j=0}^{k+2} j(j-1)a_j x^{j-1} \tag{2.3}$$

$$y''(x) = \sum_{j=0}^{k+2} j(j-1)(j-2)a_j x^{j-2} \tag{2.4}$$

From

$$y'''(x) = \sum_{j=0}^{k+2} j(j-1)(j-2)(j-3)a_j x^{j-3} \tag{2.5}$$

equations (2.1) and (2.5)

$$\sum_{j=0}^{k+2} j(j-1)(j-2)(j-3)a_j x^{j-3} = f(x, y) \tag{2.6}$$

where a_j are the parameters to be determined. Collocating (2.6) at the mesh–points $x = x_{n+j},\ j = 0(1)k$, and interpolating (2.2) at $x = x_{n+j}$, $j = 0(2)k$-1. Putting in the matrix equation form and then solved to obtain the values of parameters a_j's , j = 0, 1, which is substituted in (2.2) yields, after some manipulation, the new continuous method expressed in the form:

$$y(x) = \sum_{j=0}^{k-1} a_{2j}(x) y_{n+2j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} \tag{2.7}$$

3.0 Derivation of first- block of 5-step block method

The first-block of 5-step method for the third order initial value problem (ivp) designated by equation (1.1) can be expressed by the following matrix difference equation:

$$A^{(o)}y_q = A^{(1)}y_{q-1} + h^3B^{(0)}F_q + BF_{q-1} \quad (3.1)$$

The coefficients of continuous scheme of the 5-step block scheme are thus:

$$if \quad t = \frac{x - x_{n+1}}{h},\ a_0 = \frac{1}{8}(t^2 - 4t + 3),\ a_2 = \frac{1}{4}(-t^2 + 2t + 3),\ a_4 = \frac{1}{8}(t^2 - 1)$$

$$\beta_0 = \frac{h^3}{40320}\{-t^8 + 16t^7 - 98t^6 + 280t^5 - 336t^4 + 330t^2 - 296t + 105)$$

$$\beta_1 = \frac{h^3}{40320}\{5t^8 - 72t^7 + 350t^6 - 420t^5 - 1820t^4 + 6720t^3 - 5654t^2 - 6228t + 7119)$$

$$\beta_2 = \frac{h^3}{40320}\{-10t^8 + 128t^7 - 476t^6 - 112t^5 + 336t^4 - 13668t^2 - 16t + 10794)$$

$$\beta_3 = \frac{h^3}{40320}\{10t^8 - 112t^7 + 308t^6 + 392t^5 - 1680t^4 - 612t^2 - 280t + 1974)$$

$$\beta_4 = \frac{h^3}{40320}\{-5t^8 + 48t^7 - 98t^6 - 168t^5 + 560t^4 - 646t^2 + 120t + 189)$$

$$\beta_5 = \frac{h^3}{40320}\{t^8 - 8t^7 + 14t^6 + 28t^5 - 84t^4 + 90t^2 - 20t - 21) \quad (3.2)$$

$$y_{n+5} - \frac{15}{8}y_{n+4} + \frac{5}{4}y_{n+2} - \frac{3}{8}y_n = \frac{h^3}{384}\{f_n + 71f_{n+1} + 282f_{n+2} + 422f_{n+3} + 181f_{n+4} + 3f_{n+5}) \quad (3.3)$$

The order $P = 7$ i.e $C_{p+2} = 3.017526455 \quad \ell^{-04}$ where $f_n = f(x_n,y_n)$, $f_{n+1} = f(x_{n+1},y_{n+1})$, $f_{n+2} = f(x_{n+2},y_{n+2})$, $f_{n+3} = f(x_{n+3},y_{n+3})$, $f_{n+4} = f(x_{n+4},y_{n+4})$, $f_{n+5} = f(x_{n+5},y_{n+5})$

The first derivatives of (3.2) are, noting that

$$\frac{dt}{dx} = \frac{1}{h} \text{and} \frac{d^2t}{dx^2} = \frac{1}{h^2},$$

$$\alpha_0' = \frac{1}{8h}(2t-4), \alpha_2' = \frac{1}{4h}(-2t+2),\ \alpha_4' = \frac{1}{4h}t$$

$$\beta_0' = \frac{h^2}{40320}\{-8t^7 + 112t^6 - 588t^5 + 1400t^4 - 1344t^3 + 660t - 296\}$$

$$\beta_1' = \frac{h^2}{40320}\{40t^7 - 504t^6 + 2100t^5 - 2100t^4 - 7280t^3 + 20160t^2 - 11308t + 6228\}$$

$$\beta_2' = \frac{h^2}{40320}\{-80t^7 + 896t^6 - 2856t^5 - 560t^4 + 13440t^3 - 27336t - 16\}$$

$$\beta_3' = \frac{h^2}{40320}\{80t^7 - 784t^6 + 1848t^5 + 1960t^4 - 6720t^3 - 1224t + 280\}$$

$$\beta_4' = \frac{h^2}{40320}\{-40t^7 + 336t^6 - 588t^5 - 840t^4 + 2240t^3 - 1292t + 120)$$

The

$$\beta_5' = \frac{h^2}{40320}\{8t^7 - 56t^6 + 84t^5 + 140t^4 - 336t^3 + 180t + 20) \quad (3.4)$$

second derivative of (3.2) is:

$$\alpha_0'' = \frac{1}{4h^2}, \alpha_2'' = -\frac{1}{2h^2}, \alpha_4'' = \frac{1}{4h^2}$$

$$\beta_0'' = \frac{h}{40320}\{-56t^6 + 672t^5 - 2940t^4 + 5600t^3 - 4032t^2 + 660)$$

$$\beta_1'' = \frac{h}{40320}\{280t^6 - 3024t^5 + 10500t^4 - 8400t^3 - 21840t^2 + 40320t - 11308)$$

$$\beta_2'' = \frac{h}{40320}\{-560t^6 + 5376t^5 - 14280t^4 - 2240t^3 + 40320t - 27336)$$

$$\beta_3'' = \frac{h}{40320}\{560t^6 - 4704t^5 + 9240t^4 + 7840t^3 - 20160t - 1224)$$

$$\beta_4'' = \frac{h}{40320}\{560t^6 + 2016t^5 - 2940t^4 - 3360t^3 + 6720t^2 - 1292)$$

$$\beta_5'' = \frac{h}{40320}\{56t^6 - 336t^5 + 420t^4 + 560t^3 - 1008t + 180) \qquad (3.5)$$

4.0 Application of 5-step block method

(1) Evaluation of equation (3.2) at $x = x_{n+1}$, $x = x_{n+3}$ and $x = x_{n+5}$ yield respectively the three integrators below.

$$y_{n+5} - \frac{15}{8}y_{n+4} + \frac{5}{4}y_{n+2} - \frac{3}{8}y_n = \frac{h^3}{384}(f_n + 71f_{n+1} + 282f_{n+2} + 422f_{n+3} + 181f_{n+4} + 3f_{n+5})$$

with order $\mathrm{P} = 7$ i.e $\mathrm{C}_{p+2} = 3.017526455e^{-04}$

$$-\frac{3}{8}y_{n+4} + y_{n+3} - \frac{6}{8}y_{n+2} + \frac{1}{8}y_n = \frac{h^3}{40320}(-63f_n - 2289f_{n+1} - 10374f_{n+2} - 7434f_{n+3} + 21f_{n+4} - 21f_{n+5}) \qquad (2)$$

order $\mathrm{P} = 7$ i.e $\mathrm{C}_{p+2} = 2.686838624e^{-04}$

$$\frac{1}{8}y_{n+4} - \frac{6}{8}y_{n+2} + y_{n+1} - \frac{3}{8}y_n = \frac{h^3}{40320}(105f_n + 7119f_{n+1} + 10794f_{n+2} + 1974f_{n+3} + 189f_{n+4} - 21f_{n+5}$$

order $P = 7$ i.e $C_{p+2} = 2.521494709e^{-04})$ (4.1)

Improving the block method by considering additional equation arising from the first and second derivative functions;

$$\frac{du(x)}{dx} = Z(x), \frac{du(a)}{dx} = Z_0$$
$$\frac{d^2u(x)}{dx^2} = Z'(x), \frac{d^2u(a)}{dx^2} = Z_0' \qquad (4.2)$$

Using the first and second conditions in (1.1) on (4.2) we obtain

$$hZ_{n+5} + \frac{1}{4}y_{n+4} - y_{n+2} + \frac{3}{4}y_n = \frac{h^3}{40320}(2496 f_n + 27776 f_{n+1} + 17152 f_{n+2} + 6912 f_{n+3} - 704 f_{n+4} + 128 f_{n+5})$$

The order $P = 7$ and the error constant $C_9 = 2.716049383e^{-03}$

$$h^2Z'_{n+5} - \frac{1}{4}y_{n+4} + \frac{1}{2}y_{n+2} - \frac{1}{4}y_n = \frac{h^3}{40320}(-12640 f_n - 51264 f_{n+1} - 4992 f_{n+2} - 14720 f_{n+3} + 3552 f_{n+4} - 576 f_{n+5})$$

with order $P = 7$ and the error constant $\mathrm{C}_9 = -1.137566138e^{-02}$ (4.3)

The above discrete schemes are uniformly of order 7 and of close accuracy and could be used to start the initial value problem integration over (x_0, x_5), n = 0, 5… . Combining (4.1) and (4.4) gives the first block. There are two ways to advance the integration process after the first sub-interval {x_o, x_5}, where the block method simultaneously provides values y_1, y_2 … y_5 without recourse to any other one-step method to provide y_1. This is an improvement over the use of block method, which is not self-starting. We shall now discuss two options to move the integration process forward after the first sub-interval. First, we could proceed by using the first block 3-point in Fatunla [10] i.e block method over sub-intervals that do over-lap $(x_3, x_5\}$, $\{x_4, x_6\},\ldots,\{x_{N3}, x_N\}$. The second option is to proceed by explicitly obtaining initial conditions at x_{n+3}, n =0, 5,…,N-5, using the computed values $U(x_{n+5}) = y_{n+5}$ and $Z(x_{n+5}) = Z_{n+5}$ over sub-intervals $(x_o, x_5\},\{x_5, x_{10}\},\ldots,\{x_{N-5}, x_N\}$ which do not over-lap.

4.1 The first block to start the integration

$$\begin{bmatrix} 0 & \frac{5}{4} & 0 & -\frac{15}{8} & 1 \\ 0 & \frac{-3}{4} & 1 & \frac{-3}{8} & 0 \\ 1 & \frac{-3}{4} & 0 & \frac{1}{8} & 0 \\ 0 & -1 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{-1}{4} & 0 \end{bmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & \frac{-1}{8} \\ 0 & 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & 0 & \frac{-3}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{pmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix} +$$

$$h^3 \left[\begin{bmatrix} \frac{71}{384} & \frac{282}{384} & \frac{422}{384} & \frac{181}{384} & \frac{3}{384} \\ \frac{-2289}{40320} & \frac{-10374}{40320} & \frac{-7434}{40320} & \frac{21}{40320} & \frac{-21}{40320} \\ \frac{7119}{40320} & \frac{10794}{40320} & \frac{1974}{40320} & \frac{189}{40320} & \frac{-21}{40320} \\ \frac{27776}{40320} & \frac{17152}{40320} & \frac{6912}{40320} & \frac{-704}{40320} & \frac{128}{40320} \\ \frac{-5126}{40320} & \frac{-4992}{40320} & \frac{-14720}{40320} & \frac{3552}{40320} & \frac{-576}{40320} \end{bmatrix} \cdot \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & \frac{1}{384} \\ 0 & 0 & 0 & \frac{-63}{40320} \\ 0 & 0 & 0 & \frac{105}{40320} \\ 0 & 0 & 0 & \frac{2496}{40320} \\ 0 & 0 & 0 & \frac{-12640}{40320} \end{bmatrix} \cdot \begin{pmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix} \right] \quad (4.4)$$

5.0 Numerical experiments

This section deals with the implementation of the algorithm proposed for problems

***Problem* 5.1**

$y''' = 3\sin x$

$y(0) = 1,\ y'(0) = 0\ ,\ y''(0) = -2$

Theoretica l solution is $y(x) = 3\cos x + \frac{x^2}{2} - 2$

Table 5.1: Accuracy comparison of 5-step block method of order 7 with the same 5-step method of order 7 in predictor-corrector mode, H = 0.1

X	Exact solution *y(x)* of 5-step block Method	5-step block Method *y*-computed	Error in 5-step block Method of order 7	Error in 5-step (predictor-corrector) Method of order 7
0.1	0.99001249583	0.99001249580	3.4077519E-11	4.165922607E-09
0.2	0.96019973352	0.96019973340	1.2372514E-10	9.578013171E-08
0.3	0.91100946738	0.91100946720	1.7681812E-10	3.991507931E-07

Table 5.1: Accuracy comparison of 5-step block method of order 7 with the same 5-step method of order 7 in predictor-corrector mode, H = 0.1 (continued)

X	**Exact solution *y(x)* of 5-step block Method**	**5-step block Method *y*-computed**	**Error in 5-step block Method of order 7**	**Error in 5-step (predictor-corrector) Method of order 7**
0.4	0.84318298201	0.84318298160	4.0865533E-10	1.036855911E-08
0.5	0.75774768567	0.75774768530	3.7111825E-10	2.128500409E-06
0.6	0.65600684473	0.65600684480	7.0964790E-10	3.789530170E-06
0.7	0.53952656185	0.53952656260	7.4653450E-10	6.130076711E-06
0.8	0.41012012804	0.41012013000	1.9585035E-09	9.253855263E-06
0.9	0.26982990481	0.26982990870	3.8880070E-09	1.325713611E-05
1.0	0.12090691760	0.12090692400	6.3955807E-09	1.822776869E-05
1.1	-0.03421163572	-0.03421162620	9.5232678E-09	2.424430567E-05
1.2	-0.19292673657	-0.19292672340	1.3169979E-08	3.137524726E-05

***Problem* 5.2**

$$y''' = -y$$

$$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1$$

Theoretical solution is $y(x) = e^{-x}$

Table 5.2: Accuracy comparison of 5-step block method and 5-step (predictor-corrector) method, H = 0.1.

X	**Exact solution *y(x)* of 5-step block method**	**5-step block Method *y*-computed**	**Error in 5-step block Method of order 7**	**Error in 5-step (predictor-corrector) Method of order 7**
0.1	0.904837418	0.904837418	2.17603713E-12	1.369292790E-09
0.2	0.818730753	0.818730753	1.39354084E-11	3.122605896E-08
0.3	0.740818221	0.740818221	3.44438922E-11	1.276935889E-07
0.4	0.670320046	0.670320046	6.44767573E-11	3.251955301E-07
0.5	0.60653066	0.60653066	1.03160591E-10	6.547296992E-07
0.6	0.548811636	0.548811636	1.49799395E-10	1.144057895E-06
0.7	0.496585304	0.496585304	2.04859352E-10	1.817843959E-06
0.8	0.449328964	0.449328964	2.67552425E-10	2.697741224E-06
0.9	0.40656966	0.406568966	6938.24396E-10	3.802410064E-06
1.0	0.367879441	0.367879441	1.42294176E-10	5.147552277E-06
1.1	0.332871084	0.332871084	-7.74372511E-10	6.745807363E-06

In Tables 5.1 and 5.2, the 5-step block method of order 7 is more accurate than 5-step predictor-corrector method of the same order because the maximum absolute error for 5-step block method is 2.17603713E-12 which is significantly smaller than the maximum absolute error 1.369292790E-09 of 5-step predictor-corrector method.

***Problem* 5.3**

$$y''' = e^{x}$$

$$y(0) = 3, \ y'(0) = 1, \ y''(0) = 5$$

Theoretical solution is $y(x) = 2 + 2x^2 + e^{x}$

Table 5.3: Absolute errors for the method.

Theoretical/Approximate Solutions, h = 0.1			
X	**Exact solution *y(x)***	**5-step Block Method *y*-computed**	**Errors**
0.1	3.125170918	3.125170919	9.24352E-10
0.2	3.301402758	3.301402759	8.39834E-10
0.3	3.529858808	3.529858808	4.23997E-10
0.4	3.811824698	3.811824698	3.58729E-10
0.5	4.148721271	4.148721271	2.99872E-10
0.6	4.5421188	4.5421188	3.90509E-10
0.7	4.993752707	4.993752706	1.47048E-09
0.8	5.505540928	5.505540926	2.49247E-09
0.9	6.079603111	6.079603111	3.15695E-09
1.0	6.718281828	6.718281832	3.54096E-09
1.1	7.424166024	7.424166038	1.40536E-08
1.2	8.200116923	8.200116956	3.32635E-08

Table 5.3 shows the exact solutions $y(x)$, y-computed and errors for problem 3, the maximum absolute error is observed to be 9.24352E-10. The low value of the error shows that the new method is accurate.

***Problem* 5.4**

$$y''' = -e^x$$

$$y(0) = 1,\ y'(0) = -1,\ y''(0) = 3$$

Theoretical solution is $y(x) = 2 + 2x^2 - e^x$

Table 5.4: Absolute errors for the method.

Theoretical/Approximate Solutions, h = 0.1			
X	**Exact solution *y*(*x*)**	**5-stepBlock Method *y*-computed**	**Errors**
0.1	0.914829082	0.914829082	7.56477E-11
0.2	0.858597242	0.858597242	2.6017E-10
0.3	0.830141192	0.830141193	5.76003E-10
0.4	0.828175302	0.828175303	8.4127E-10
0.5	0.851278729	0.85127873	1.00013E-09
0.6	0.8978812	0.897881201	1.09051E-09
0.7	0.966247293	0.966247294	1.07048E-09
0.8	1.054459072	1.054459073	1.49247E-09
0.9	1.160396889	1.160396892	3.15695E-09
1.0	1.281718172	1.281718176	4.45905E-09
1.1	1.415833976	1.415833983	6.94643E-09
1.2	1.559883077	1.559883087	9.73655E-09

In Tables 5.4, the exact solutions *y(x)*, *y*-computed and errors for Problem 5.4 is shown, the maximum absolute error is observed to be 7.56477E-11 . The low value of the error shows that the new method is accurate.

6.0 Conclusion

The new 5-step block method of order 7 proposed in this paper is more accurate than when the same 5-step scheme of the same order 7 is applied over overlapping intervals in predictor-corrector mode as shown in Tables 5.1 and 5.2, in which the maximum absolute error for 5-step block method is significantly smaller than that of 5-step predictor-corrector method. The block solution form is in fact a remarkable achievement as it eliminates the use of predictors. In predictor –corrector method, we get predictors which are of lower order, to combine with high order corrector thereby reducing the overall accuracy. The block method is also more accurate because the maximum absolute error for block method is significantly smaller than that of predictor-corrector method. It is faster in the sense that it is applied as simultaneous integrator for the special third order ordinary differential equations.

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