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An approach for solving linear fractional programming problems

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Abstract

Linear fractional programming problems are useful tools in production planning, financial and corporate planning, health care and hospital planning and as such have attracted considerable research interest. The paper presents a new approach for solving a fractional linear programming problem in which the objective function is a linear fractional function, while the constraint functions are in the form of linear inequalities. The approach adopted is based mainly upon solving the problem algebraically using the concept of duality and partial fractions and an example is given to clarify the developed method.

Key words
Linear Fractional Programming; Linear Fractional Function, Duality Concept, Objective Function.

1.0 Introduction.

Linear programming is a mathematical technique aimed at identifying optimal maximum or minimum values of a problem subject to certain constraints Odior, [1], while a linear fractional programming (LFP) problem is the one whose objective function has a numerator and a denominator and are very useful in production planning, financial and corporate planning, health care and hospital planning. Several methods to solve this problem have been proposed Tantawy, [2]. Charnes and Kooper [3] have proposed a method which depends on transforming the LFP problem to an equivalent linear program . Another method is called up dated objective function method derived by Bitran and Novaes [4] which is used to solve the linear fractional program by solving a sequence of linear programs only re-computing the local gradient of the objective function. Also some aspects concerning duality and sensitivity analysis in linear fraction program was discussed by Bitran and Magnant [5].

This kind of mathematical programming problem has attracted considerable research and interest, since they are useful in production planning, financial planning corporate planning and health care planning. However for a single objective linear fractional programming the Charnes and Cooper [6] transformation can be used to transform the problem into a linear programming problem. Few approaches have been reported for solving the multiple objective linear fractional programming (MOLFP) problem. Kormbluth and Steuer [7] considered this problem and presented a simplex –based solution procedure to find all weakly efficient vertices of the augmented feasible region. Also Benson [8] in his article showed that the procedure

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suggested by Kormbluth and Steuer [7] for computing the numbers to find break points may not work all the time and he proposed a fail safe method for computing these numbers. The objective space for multiple objectives linear fractional programming with equal denominators was given by Tantawy [2], using the concept of duality. The approach in this study will enable the transformation of a single objective linear fractional programming problem into a linear programming problem using partial fractions method with the concept of duality.

2.0 Definition of a linear fractional programming problem

A linear fractional programming problem occurs when a linear fractional function is to be minimized or maximized and the problem can be formulated mathematically as follows: A linear - fractional programming problem is of the type;

Maximize P(x) = $\dfrac{c^T x + \alpha}{f^T x + \beta}$, subject to

$$x \in X = (x, Ax \leq b) \qquad (2.1)$$

$f^T x + \beta \geq 0$, $x \geq 0$, where x $\in$ R^n , A is an $(m + n) \times n$ matrix, c and d are n-vectors, $b \in R^{m+n}$, and α , β are scalars. It is assumed that the feasible solution set X is bounded and closed (compact set). Assume $\lim_{h \to 0} \dfrac{\alpha}{h} = +\infty$ if $\alpha > 0$ and $\lim_{h \to 0} \dfrac{\alpha}{h} = -\infty$ if $\alpha \leq 0$. Generalizing a linear–fractional programming problem will give,

$$\text{maximize } F(x) = \frac{c_i^T x - d_i}{f_i^T + g_i} \qquad (2.2)$$

subject to $Ax \leq b$, $f_i^T x + g_i \geq 0$, $i = 1, \ldots, k$.

$$\text{maximize } F(x) = (c_i - \frac{d_i}{g_i} f^T)\frac{x}{f^T x + g} + \frac{d_i}{g_i}$$

$$\text{subject to } (A + \frac{d_i}{g_i} f^T)\frac{x}{f^T x + g} \leq \frac{b_i}{g_i} \qquad (2.3)$$

Defining $\dfrac{x}{f_i^T + g_i} \geq 0$, then (2.3) can be written in the form

$$\text{Maximize F(y)} = (c_i - \frac{d_i}{g_i} f^T)y + \frac{d_i}{g_i}$$

$$\text{subject to } (A + \frac{d_i}{g_i} f^T)y \leq \frac{b_i}{g_i} \qquad (2.4)$$

Equation 2.4 can simply be written in the form

$$\text{Maximize } F(y) = p^T y + \frac{b_i}{g_i}$$

$$\text{Subject to } Gy \leq t \qquad (2.5)$$

where $p^T = (c_i - \dfrac{d_i}{g_i} f^T)$, $G = (A + \dfrac{d_i}{g_i} f^T)$ and $t = \dfrac{b_i}{g_i}$. From (2.4) which defines y, we have

$$x = g_i \frac{y}{1 - f^T y} \qquad (2.6)$$

Now, consider the dual of linear programming in (2.5) in the form

$$\text{Minimize } w = u^T t$$
$$\text{Subject to } u^T G = p^T \qquad (2'7)$$
$$u \geq 0.$$

On multiplying the set of constraints of this dual problem by $T = (T_1 \mid T_2)$, $T_1 = p(p^T p)^{-1}$, the column of the matrix T_2 constitutes the base of $N(p^T) = \{v;\ p^T v = 0\}$ (2.8)

We have $u^T GT_1 = 1$, $u^T GT_2 = 0$ and $u \geq 0$. In this case when $GT_2 \neq 0$, an $s \times (m + n)$ matrix Q of non-negative entries is defined such that $QGT = 0$. This matrix will play an important role to find the optimal value of the above problem as the maximum value of w on the interval on the real line defined by $W = \{w \in R \mid QGT_1 w \leq Q\, g\}$. The above representation can simply be written as $W = \{w \in R \mid Zw \leq z\}$, where

$Z = QGT_1$ and $z = Q\, g$ (2.9)

Also a sub matrix $\bar{Q}$ of the given matrix Q satisfying $\bar{Q}GT_1 = 1$ will be important for specifying the dual values needed for solving the linear fraction programming problem (2.1) . The dual values satisfy the well known Kuhn-Takucer condition Tantawy, [2] and for a point y^k to be an optimal solution of the above program, (2.5) must exist.

$u \geq 0$ such that $G^T u = p$, or simply $u = (G, G^T,)^{-1} G, p$ (2.10)

3.0 New method for solving linear fractional programming (lfp) problems.

The new method for solving LFP problems is summarized as follows:

(1) Compute $T_1 = (p^T p)^{-1}$p, and the matrix T_2 as in (2.8)

(2) Find the matrix Q of non-negative entries such that $QGT_2 = 0$,

(3) Find a sub matrix $\bar{Q}$ of the given matrix Q satisfying $\bar{Q}GT_1 = 1$

(4) In the rows of $\bar{Q}$ for every positive entry, determine the corresponding active constraint in the given matrix GT_1

(5) Solve an $n \times n$ system of linear equations for these set of active constraints to get the optimal solution y^*. Then use (2.6) to get the optimal solution of the (LFP) problem defined by (2.1).

***Remarks* 3.1**

(1) The matrix Q of non-negative entries such that QGT = 0, is considered as the a polar matrix of the given matrix GT_2

(2) With $d = 0$ in (LFP), the above problem reduces to linear programming problem (LP), and hence the method can be used to solve the (LP) as a special case of this (LFP) using the same argument.

3.1 Numerical Examples

***Example* 3.1.**

Maximize $Z = \dfrac{x_1 + 3x_3 + 2x_3}{2x_1 + x_2 + 4x_3 + 1}$, subject to $x_1 + 3x_2 + 6x_3 \leq 8$, $2x_1 + x_2 + 4x_3 \leq 5$, $x_1, x_2, x_3 \geq 0$

Table 3.1*: The final table for $x_1 = x_3 = 0$, $x_2 = 8/3$ to be an optimal solution.*

a_1	a_2	a_3	a_4	a_5	X_B
1/3	1	2	1/3	0	8/3
5/3	0	2	1/3	1	7/3
40/3		92/3	1		

For the solution $x_2 = 8/3$, $x_1 = x_3 = 0$, we observe that first constraint $\frac{1}{3}x_1 + x_2 + 2x_3 \leq \frac{8}{3}$ holds as an equality, while the second constraint $\frac{5}{3}x_1 + 0x_2 + 2x_3 \leq \frac{7}{3}$ holds as an inequality.

This second constraint is an invalid constraint. We therefore combine this invalid constraint with the objective function to generate the following parametric fractional programming problem. $Z = \dfrac{x_1 + 3x_2 + 2x_3 + \lambda(\frac{5}{3}x_1 + 2x_3)}{2x_1 + x_2 + 4x_3 + 1 + \mu(\frac{5}{3}x_1 + 2x_3)}$, subject to $\frac{1}{3}x_1 + x_2 + 2x_3 = \frac{8}{3}$, $x_1, x_2, x_3 \geq 0$.

***Example* 3.2**

Consider the following combination of linear fractional programming problems (LFP),

Maximize $z_1 = \dfrac{x_1 + x_3 + 2}{x_1 + 2}$ and Maximize $z_2 = \dfrac{-x_1 + 2x_2 + 4}{x_1 + 2}$, subject to $x_1 + x_2 + x_3 \leq 1$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. Using partial fractions with duality concept we have,

$$c_1^t = (1\ \ 0\ \ 1),\ c_2^t = (1\ \ 2\ \ 0),\ d^t = (1\ \ 0\ \ 0), \text{ also } \alpha_1 = 2,\ \alpha_2 = 4 \text{ and } \beta = 2$$

***Example* 3.3**

Consider the following linear fractional programming problem (LFP), Maximize $z = \dfrac{x_1 + x_2 + 3}{x_2 + 1}$, subject to $x_1 + 2x_2 \leq 6$, $-x_1 \leq 0$, $-x_2 \leq 0$. For this LFP we have $c^t = (1\ 1)$, $d^t = (0\ 1)$, $b_i = 3$, $g_i = 1$, (c and d are matrices) and then we have

$$T_1 = \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix},\ T_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } GT = \begin{pmatrix} 10 \\ -2 \\ -1 \end{pmatrix} \text{ which gives } Q = \begin{pmatrix} 1 & 5 & 0 \\ 1 & 0 & 10 \end{pmatrix}$$

The second row in Q satisfies $\bar{Q}GT_1 = 1$. This indicates that the first and the third constraints in G are the only active set of constraints, on solving $y_1 + 8y_2 = 6$, $y_2 = 0$, we get $y^{t*} = (6\ \ 0)$ as the optimal solution for the equivalent problem which finally on using (2.6) gives $x^{t*} = (6\ \ 0)$ as the optimal solution of our linear fraction program with optimal value $z^* = 9$.

4.0 Conclusion

A method for solving linear fractional functions with constraint functions in the form of linear inequalities is given. The proposed method differs from the earlier methods as it is based upon solving the problem algebraically using the concept of duality with partial fractions approach. The method appears simple to solve any linear fractional programming problem of any size.

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