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On the statistical properties of the non-linear water waves

Ifeoma O. Ejinkonye
Department of Mathematics and Computer Science
Nigerian Defence Academy
Kaduna, Nigeria

Abstract

The study of the statistical properties of the non-linear random wave had been earlier investigated. In this work we introduce a bi-parametric distribution of non-linear stochastic processes, in studying the properties of second-order random processes with a narrow-band spectrum. This incidentally concerns the mechanics of the water waves. In particular, the expressions of the probability density function are further investigated, using this bi-parameter. This analysis will enable the designer to choose wave parameters, within a limit, that will yield an acceptable level of risk. Secondly, such probabilistic based design criterion may result in substantial cost saving if uncertainties in the wave estimates are incorporated.

Keywords
Stochastic processes, narrow-band spectrum, probability density function, bi-parametric distribution.

1.0 Introduction

The effect of non-linearity for random (wind-generated) sea waves were firstly investigated by Longuet-Higgins [1]. He obtained the first three terms of Gram-Charlier series for the probability density function of the normalized free surface displacement. This was found correct for any shape of the energy spectrum.

After, Tayfun [2, 3] obtained the probability density function which explained the exceedances of the crest (absolute maximum) for the free surface displacement in an, undisturbed wave field. The corresponding probability of the exceedance of the trough (absolute minimum) was then derived by Tung and Huang [4]. Arena and Fedele [5] extended the theory to the crest and the trough distributions of a general nonlinear narrow-band stochastic family, which involves many process in the mechanics of the sea waves.

In this work a new theoretical approach is proposed to investigate the effects of non-linearity in the studying of the mechanics of sea waves. In this case, a bi-parametric distribution of non-linear stochastic processes is introduced.

Some properties of the stochastic distribution are derived. Further, the characteristics function was obtained using the Laplace transformation method and the corresponding probability density function was given by the inverse Fourier transform of the characteristics function.

Telephone: 08060297345

2.0 The properties of a stochastic distribution with narrow-band spectrum

We use the coordinator axis (*x*,*y*) in which *x*-axis is perpendicular to the wave crest and *y*-axis perpendicular to its origin being on the shoreline.

We define the distribution ψ of stochastic processes axis (*x*, *y*) as parameters following (Arena and Fedele [5].

$$\psi(x,y,t) = f(x,y)a\cos[X(t)] + g(x,y)a^2\cos^2[X(t)] + h(x,y)a^2\sin^2[X(t)], \quad (2.1)$$

where a is stochastic variable with Rayleigh Distribution and where

$$X(t) = \omega_0 t + \upsilon \quad (2.2)$$

ω_0 is the angular frequency, *t* the time and υ a stochastic variable uniformly distributed in (0, 2π). By defining the two stochastic processes:

$$Z_1 = \frac{a\cos(X)}{\sigma}, \quad Z_2 = \frac{a\sin(X)}{\sigma} \quad (2.3)$$

where σ^2 is the variance of both the linear processes *acos(X*) and *a*sin(*X*), equation (2.1) may be written as:

$$\psi(Z_1, Z_2) = \sigma\left[F(x,y)Z_1 + G(x,y)Z_1^2 + H(x,y)Z_2^2\right], \quad (2.4)$$

where

$$\left.\begin{aligned} F(x,y) &\equiv f(x,y), \\ G(x,y) &\equiv \sigma g(x,y), \\ H(x,y) &\equiv \sigma h(x,y) \end{aligned}\right\} \quad (2.5)$$

The processes (Z_1,Z_2) are both Gaussian (with zero mean value and unitary variance) and stochastically independently Borgman [6]. Therefore, the joint probability density function is given by

$$F_{Z_1Z_2}(Z_1, Z_2) = \frac{1}{2\pi}e^{-\frac{1}{2}\left(Z_1^2 + Z_2^2\right)} \quad (2.6)$$

From equation (2.4) we obtain the mean value and the variance of ψ which are respectively given by

$$\overline{\psi} = \sigma(G + H), \quad (2.7)$$

$$\sigma_\psi^2 = \frac{\sigma^2 F^2}{\beta^2}, \quad (2.8)$$

where

$$\beta = \frac{1}{\sqrt{1 + 2\left(\alpha_1^2 + \alpha_2^2\right)}} \quad (2.9)$$

$$\alpha_1 = \frac{G(x,y)}{|F(x,y)|}, \quad \alpha_2 = \frac{H(x,y)}{|F((x,y))|} \quad (2.10)$$

Finally, we may consider the following normal stochastic distribution defined as:

$$\zeta = \frac{\psi - \overline{\psi}}{\sigma_\psi} = \beta\left(Z + \alpha_1 Z_1^2 + \alpha_2 Z_2^2\right) - \beta(\alpha_1 + \alpha_2), \quad (2.11)$$

in which α_1 and α_2 are deterministic parameters. The properties of the (2.11) rely on these two parameters. As an example and following Ejinkonye [7] analytical expressions of the third and fourth moments of the family are given respectively by:

$$\overline{\zeta^3} = \beta^3\left[6\alpha_1 + 8\alpha_1^3 + 8\alpha_2^3\right], \quad (2.12)$$

$$\overline{\zeta^4} = 3\beta^3\left(1 + 20\alpha_1^2 + 4\alpha_2^2 + 20\alpha_1^4 + 8\alpha_1^2\alpha_2^2 + 20\alpha_1^4\right), \qquad (2.13)$$

3.0 The probability density function

Let us consider the normalized distribution ζ [equation 2.13]. The characteristics function of ζ is equal to the mean value of $e^{i\omega\zeta}$

$$\overline{e^{i\omega\zeta}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega\zeta} F_{Z_1Z_2}(z_1, z_2)dz_1dz_2, \qquad (3.1)$$

and may be rewritten as following Arena et al [5] as

$$\overline{e^{i\omega\zeta}} = \frac{1}{2\pi}\exp\left[-i\omega\beta\left(\alpha_1 + \alpha_2\right)\right]I_1I_2, \qquad (3.2)$$

with the integrals I_1 and I_2 respectively defined as:

$$I_1(w;\alpha_1,\alpha_2) = 2\int_0^{\infty}\cos(w\beta z_1)\exp\left[-\frac{z_1^2}{2}(1 - 2iw\beta\alpha_1)\right]dz_1 \qquad (3.3)$$

$$I_2(w;\alpha_1,\alpha_2) = 2\int_0^{\infty}\exp\left(iw\beta\alpha_2 z_2^2\right)\exp\left(-\frac{1}{2}z_2^2\right)dz_2 \qquad (3.4)$$

The integrals I_1 and I_2 are evaluated by using Laplace transform method. In particular, defining $z_1^2 = t$ and $z_2^2 = t$, the integrals (3.3) and (3.4) are respectively given by

$$I_1 = \int_0^{\infty}\exp\left[-\frac{t}{2}(1 - 2iw\beta\alpha_1)\right]\frac{\cos\left(w\beta\sqrt{t}\right)}{\sqrt{t}}dt = L\left\{\left(\frac{\cos\left(w\beta\sqrt{t}\right)}{\sqrt{t}}\right), S = \frac{1 - 2iw\beta\alpha_1}{2}\right\} \qquad (3.5)$$

$$I_2 = \int_0^{\infty}\exp\left(-\frac{t}{2}\right)\frac{\exp(iw\beta\alpha_2 t)}{\sqrt{t}}dt = L\left\{\frac{\exp(iw\beta\alpha_2 t)}{\sqrt{t}}, S = \frac{1}{2}\right\} \qquad (3.6)$$

where

$$L[\mathbf{g}(t), s] = \int_0^{+\infty} e^{-st}g(t)dt, \ \ \mathrm{S} > 0 \qquad (3.7)$$

defines the Laplace transform of $g(t)$:

The Laplace transforms in equations (3.5) and (3.6) becomes respectively:

$$L\left(\frac{\cos\left(\lambda\sqrt{t}\right)}{\sqrt{t}}, s\right) = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-\frac{\lambda^2}{4s}} \qquad (3.8)$$

$$L\left(\frac{e^{\lambda t}}{\sqrt{t}}, s\right) = L\left(\frac{1}{\sqrt{\mathrm{t}}}, s - \lambda\right) = \frac{\sqrt{\pi}}{\sqrt{s - \lambda}} \qquad (3.9)$$

and the characteristics function (3.2) is given by:

$$\ell^{\overline{iw\zeta}} = \frac{\exp\left[-\frac{1}{2}\frac{(w\beta)^2}{1 + 4(w\beta\alpha_1)^2}\right]\exp\left\{-iw\beta(\alpha_1 + \alpha_2) + \frac{(w\beta)^2\alpha_1}{1 + 4(w\beta\alpha_1)^2}\right\}}{\sqrt{1 - 4(w\beta)^2\alpha_1\alpha_2 - 2iw\beta(\alpha_1 + \alpha_2)}} \qquad (3.10)$$

Finally, the probability density function f_ζ is obtained by applying the inverse Fourier transform to the characteristic function $\ell^{iw\zeta}$, that is

$$f_\zeta(\zeta) = F^{-1}\left(\ell^{\overline{iw\zeta}}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ell^{iw\zeta}\,\ell^{\overline{iw\zeta}}\,dw \tag{3.11}$$

jn which F^{-1} is the inverse Fourier transform operator, define as

$$F^{-1}[f(w), x] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ell^{-iwx} f(w)\,dw \tag{3.12}$$

from equation (3.10) and (3.11) we obtain the general expression of the probability density function f_ζ.

$$f_\zeta(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \ell^{-iw\zeta} \frac{\exp\left[-\frac{1}{2}\frac{(w\beta)^2}{1+4(w\beta\alpha_1)^2}\right]\exp\left\{-iw\beta(\alpha_1+\alpha_2)+\frac{(w\beta)^2\alpha_1}{1+4(w\beta\alpha_1)^2}\right\}}{\sqrt{1-4(w\beta)^2\alpha_1\alpha_2-2iw\beta(\alpha_1+\alpha_2)}}dw \tag{3.13}$$

4.0 The zero-mean value processes

The stochastic distribution (2.1) has zero mean value if G + H = 0. In this case expression (2.4) may be rewritten as

$$\psi = \sigma\left[FZ_1 + G\left(Z^2 - Z_2^2\right)\right] \tag{4. 1}$$

And the dimensionless process ζ in (2.11) may be rewritten as

$$\zeta = \beta\left[Z_1 + \alpha\left(Z_1^2 - Z_2^2\right)\right] \tag{4.2}$$

where

$$\alpha = \frac{G(x,y)}{|F(x,y)|} \text{ and } \beta = \frac{1}{\sqrt{1+4\alpha^2}}$$

Let us note that G+H = 0 implies $\alpha_1 = -\alpha_2$ of equation (2.10). The distribution with zero –mean value has then only one parameter. The expressions (2.13) of the third moment and (2.14) of the fourth moment become as the following

$$\overline{\zeta^3} = 6\beta^3\alpha, \quad \overline{\zeta^4} = 3\beta^4\left(1 + 24\alpha^2 + 48\alpha^4\right) \tag{4.3}$$

Finally, the probability density function (3.13) for the zero-mean processes reduces itself to

$$f_\xi(\zeta) = \frac{1}{\pi}\int_0^{+\infty} \frac{\ell^{-\frac{1}{2}\frac{(w\beta)^2}{1+4(w\beta\alpha)^2}}}{\sqrt{1+4(w\beta\alpha)^2}} \cos\left[w\left(\zeta + \beta\frac{(w\beta)^2\alpha}{1+4(w\beta\alpha)^2}\right)\right]dw \tag{4.4}$$

Figure 4.1 shows the probability density function (4.4), for fixed value of the parameter α. Let us note that the probability density function (4.4) tends to the Gaussian distribution when $\alpha \to 0$.

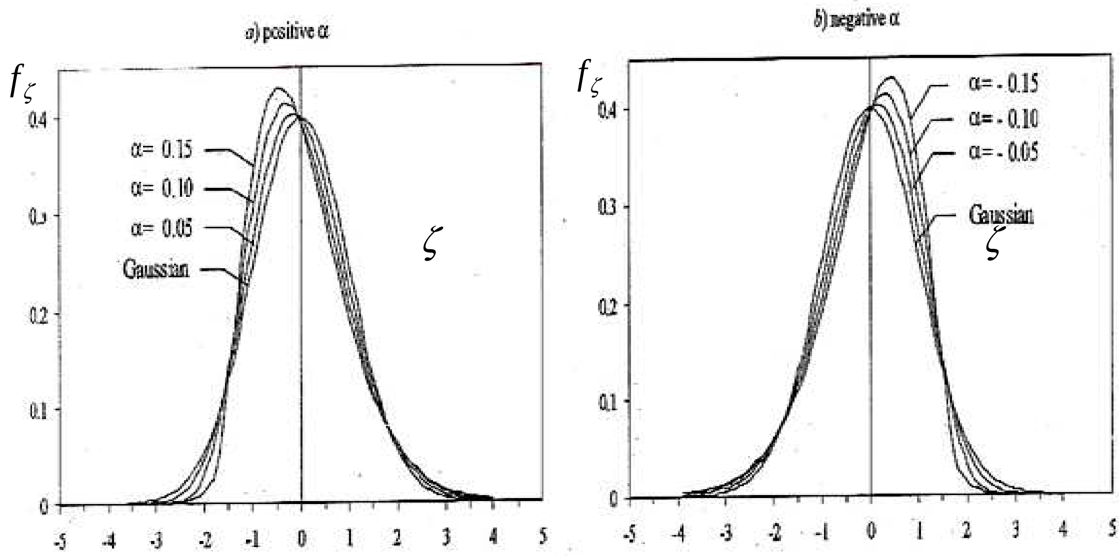

Figure 4.1: The probability density function f_ς (equation (4.4), for fixed values of α . The f_ς tends to the Gaussian distribution as $\alpha \to 0$.

5.0 Conclusion

The properties of the distribution $\psi(x,y,t) = f(x,y)a\cos[X(t)] + g(x,y)a^2\cos^2[X(t)] + h(x,y)a^2\sin^2[X(t)]$ have been investigated. For this purpose the analytical expressions of the probability density function were derived.

It is illustrated that all these properties depend upon two deterministic parameters named α_1 and α_2. For zero mean Gaussian processes we have $\alpha_1 = -\alpha_2$ and the distribution has only one degree of freedom.

These bi-parameters will be useful in solving some problems of random ocean waves. First, it allows the designer to choose wave parameters within a limit that will yield an acceptable level of risk. Second, such probabilistic-based design criterion may result in substantial cost saving if uncertainties in the wave estimates are incorporated.

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