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Complete controllability of perturbed infinite delay systems

[1]Davies Iyai and [2]Reuben A. Umana
[1]Department of Mathematics and Computer Science,
Rivers State University of Science and Technology,
P. M. B 5080, Port Harcourt, Rivers State, Nigeria.
[2]Department of Mathematics,
Federal University of Technology, Owerri, Nigeria

Abstract

Sufficient conditions for the complete controllability of perturbed infinite delay systems are eveloped. The results are established using the Schauder's fixed point theorem. An example is also given.

***Keywords*:** Delay systems, perturbation, complete controllability, properness

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1.0 Introduction

The theory of functional differential equation have been studied by several authors, for example Driver [8], Hale [9], Tadmore [14] etc., and independent results obtained. The study of functional differential equation has application in population dynamics, conveyor belts, metal rolling system, urban traffic, and capacity management.

The control equations of linear and nonlinear functional differential equations have applications in some economic and physiological systems, as well as electromagnetic systems composed of subsystems interconnected by hydraulic and various other linkages. Motivation for such control systems and its application in other fields can be found in Iheagwam [10], Chukwu [4], Davies [6-7], Niamsup and Phat [12] etc.

Owing to difficulty that arises in presenting real life situation in ecology, epidemics, population growth etc., the study of integro-differential equation with infinite delay has emerged as a branch of modern research (see Burton [2], Corduneanu [5], and Lakshmikantham [11] for details). For example, in most biological populations the accumulation of metabolic products may seriously inconvenience a population, and one of the consequences can be a fall in the birth rate and an increase in the mortality rate. If it is assumed that the total toxic action on birth and death rates is expressed by an integral term in the logistic equation, then an appropriate model is an integro-differential equation with infinite delay (Balachandran and Dauer [1]). These studies have been extended to the controllability of infinite functional differential equations in recent years. In [1], Balachandran and Dauer gave sufficient condition for the null controllability of nonlinear infinite delay systems with time varying multiple delays in control. Sinha [13], developed sufficient conditions for the null controllability of nonlinear infinite delay systems with restrained controls. Davies [6] proved sufficient condition for the Euclidean controllability of infinite delay systems with limited controls.

[1]Corresponding author, e-mail: davsdone@yahoo.com, Phone: 08054335845

Motivated by the works in [1, 6, 13] we shall forge ahead to investigate the complete controllability of perturbed infinite delay systems by the notion of linearization and the application of Schauder's fixed point theorem. This will extend the work in Davies [6], and other known result in the subject

2.0 Basic notations and preliminaries

Let E denote the real line and $J=[t_0,t_1]$ an interval in E. For a positive integer n, we denote by E^n, the space of real n-tupples with the Euclidean norm denoted by $|\cdot|$. Let $\gamma \geq h \geq 0$ be given real numbers (γ may be $+\infty$). The function $\eta:[-\gamma,0]\to[0,\infty)$ is lebesgue integrable on $[-\gamma,0]$, positive and non-decreasing on $[-\gamma,0]$. Let $B=B([-\gamma,0],E^n)$ be the Banach space of functions which are continuous and bounded on $[-\gamma,0]$, and such that $|\phi|=\sup_{\theta\in[-h,0]}|\phi(\theta)|+\int_{-\gamma}^{0}\eta(\theta)\phi(\theta)d\theta<\infty$, for any $x:[t-\gamma,t]\to E^n$, let $x_t:[-\gamma,0]\to E^n$ be defined by $x_t(\theta)=x(t+\theta),\ \theta\in[-\gamma,0]$. Let $W_2^{(1)}$ denote the Sobolev space $W_2^{(1)}([-h,0],E^n)$ of functions $\phi:[-h,0]\to E^n$ whose derivative are square integrable. We consider the infinite delay system given as

$$\dot{x}(t)=L(t,x_t)+Cu(t)+\int_{-\infty}^{0}A(\theta)x(t+\theta)d\theta \tag{2.1}$$

and its perturbation

$$\begin{aligned}\dot{x}(t)&=L(t,x_t)+Cu(t)+\int_{-\infty}^{0}A(\theta)x(t+\theta)d\theta\\ &\quad+f(t,x(t),x(t-1),x(t-2),u(t),u(t-h))\\ x(t)&=\phi(t),\quad t\in(-\infty,0]\end{aligned} \tag{2.2}$$

where

$$L(t,\phi)=\sum_{K=0}^{N}A_k\,\phi(-t_k) \tag{2.3}$$

satisfied almost everywhere on $[t_0,t_1]$. $L(t,\phi)$ is continuous in t, linear in ϕ. A_k is a continuous $n\times n$ matrix function for $0\leq t_k\leq\tau$, $A(\theta)$ is an $n\times n$ matrix whose elements are square integrable on $(-\infty,0]$ and C is an $n\times m$ matrix function. The n-vector function f is continuous and absolutely continuous.

The controls u are square integrable with values in the unit cube $C^m=\{u\in E^m:|u_j|\leq 1,\quad j=1......m\}$

The variation of constant formula for system (2.1) by Davies [6] and all its necessary assumption is

$$x(t)=X(t)\phi(0)+\int_{t_0}^{t}X(t-s)Cu(s)ds+\int_{t_0}^{t}X(t-s)\int_{-\infty}^{t}A(\theta)x(t+\theta)d\theta ds \tag{2.4}$$

The corresponding result for (2.2) at $t = t_1$ following the methods Sinha [13] is given by

$$x(t_1, \phi, f) = X(t)\phi(0) + \int_{t_0}^{t_1} X(t-s)C\,u(s)ds$$
$$+ \int_{t_0}^{t_1} X(t-s)\int_{-\infty}^{0} A(\theta)x(t+\theta)d\theta ds \qquad (2.5)$$
$$+ \int_{t_0}^{t_1} X(t-s)\,f(s, x(s), x(s-1), x(s-2), u(s), u(s-h))ds$$

Define $Y(s,t) = X(t-s)C$, and the controllability matrix by $W(t_1) = \int_{t_0}^{t_1} Y(t,s)Y^T(t,s)\,ds$ where T denotes the matrix transpose.

***Definition* 2.1**

System (2.2) is completely controllable if for every $\phi \in W_2^{(1)},\ x_1 \in E^n,$ there exists a $t_1 > t_0$ and an admissible control u such that the solution $x(t_1, \phi, f)$ of (2.2) satisfies $x(t_0, \phi, f) = \phi$ and $x(t_1, \phi, f) = x_1$

3.0 Controllability results

Here, we give theorems which will summarize our results on relative controllability of the system (2.1). Following the methods of Chukwu [3] we introduce the "determining equations" for the system

$$\dot{x}(t) = \sum_{k=0}^{N} A_k x(t - t_k) + C\,u(t) \qquad (3.1)$$

where $0 = t_0 < t_1$ and $A_0,\ A_1$ and C are constant matrices, as

$$Q_k(s) = A_0\,Q_{k-1}(s) + A_1\,Q_{k-1}(s-\tau)$$
$$k = 1, 2.....\ \ s \in (-\infty, \infty),\quad Q_0(s) = 0,\quad s \neq 0 \qquad (3.2)$$

and define $\hat{Q}_n(t_1) = \{Q_0(s), Q_1(s), ..., Q_{n-1}(s);\ s \in [0, t_1)\}$

***Proposition* 3.1**

The control system (2.1) is relatively controllable in E^n on the interval $[t_0, t_1]$ if and only if $rank\ \hat{Q}_n(t_1) = n.$

***Remark* 3.1**

To prove the above Proposition, we use the fact that $Y(t,s) = X(t-s)C$ and proceeds as in Theorem 2.1 of Davies [6] and the references therein.

***Proposition* 3.2**

System (2.1) is proper on $[t_0, t_1],\ t_1 > t_0$ if and only if the origin is an interior point of $R(t_1)$.

Proof

This is Theorem 2.3 of Davies [6]

4.0 Main result

Here, we are now ready to obtain our main result on the complete controllability of the nonlinear delay system (2.2). For this, we take

$$p=(x_0,x_1,x_2,u_0,u_1)\in E^n\times E^n\times E^n\times E^m\times E^m$$

and let $|p|=|x_0|+|x_1|+|x_2|+|u_0|+|u_1|$

***Theorem* 4.1**

In (2.2) assume;

(i) System (2.1) is relatively controllable on $[t_0,t_1]$

(ii) The continuous function f satisfies the growth condition

$$\lim_{|p|\to\infty}\frac{|f(t,p)|}{|p|}=0$$

uniformly in $t\in[t_0,t_1]$, then system (2.2) is completely controllable on $[t_0,t_1]$.

Proof

From equation (2.5), the solution of system (2.2) can be rewritten as

$$x(t_1,\phi,f)=x_L(t_1)+\int_{t_0}^{t_1}Y(t,s)u(s)\,ds$$
$$+\int_{t_0}^{t_1}X(t-s)f(s,x(s),x(s-1),x(s-2),u(s),u(s-h))ds$$

where $$x_L(t_1)=X(t)\phi(0)+\int_{t_0}^{t_1}X(t-s)\int_{-\infty}^{0}A(\theta)x(t+\theta)d\theta\,ds$$

let B be the Banach space of all functions $(x,u):[t_0,t_1]\to E^n\times E^m$ where x is continuous and u is an admissible control function. The norm on B is $\|(x,u)\|=\|x\|+\|u\|$ where

$$\|x\|=\sup|x(t)|\text{ for }t\in[t_0,t_1]$$
$$\|u\|=\sup|u(t)|\text{ for }t\in[-h,t_1]$$

Let $T:B\to B$ be an operator defined by $T(x,u)=(y,v)$ where

$$v(t)=Y^T(t,s)W^{-1}(t_1)\times$$
$$\left[x_1-x_L(t_1)-\int_{t_0}^{t_1}X(t-s)f(s,x(s),x(s-1),x(s-2),u(s),u(s-h))\,ds\right]\text{for}$$

$t\in[t_0,t_1]$, and $v(t)=0$ for $t\in[-h,0]$, $x_1\in E^m$;

$$y(t)=x_L(t_1)+\int_{t_0}^{t_1}Y(t,s)v(s)\,ds$$
$$+\int_{t_0}^{t_1}X(t-s)f(s,x(s),x(s-1),x(s-2),u(s),u(s-h))\,ds$$

for $t\in[t_0,t_1]$, and $y(t)=\phi(t)$ for $t\in[-h,0]$.

Observe that the control $v(t)$ is capable of steering the solution of system (2.2) to x_1 at $t=t_1$.

Let $a_1=\sup|Y(t,s)|$ *for* $t_0\le t\le t_1$, $a_2=|W^{-1}(t_1)|$,

$a_3=\sup|x_L(t_1)|+|x_1|$ *for* $t_0\le t\le t_1$

$a_4=\sup|X(t-s)|$ *for* $(t-s)\in[t_0,t_1]\times[t_0,t_1]$

$b = \max\{t_1, a_1, h\}$, $c_1 = 10ba_1a_2a_4t_1$, $c_2 = 10a_4t_1$, $d_1 = 10a_1a_2a_3b$

$d_2 = 10a_3$, $c = \max\{c_1, c_2\}$, $d = \max\{d_1, d_2\}$

Then $|v(t)| \le a_1a_2\left[a_3 + a_4t_1\left(\sup|f(t,x(t),x(t-1),x(t-2),u(t),u(t-h))|\right) \quad for \quad t\in[t_0,t_1]\right]$

$$= \frac{d_1}{10b} + \frac{c_1}{10b}\left(\sup|\, f(t,x(t),x(t-1),x(t-2),u(t),u(t-h)|\right) \; for \; t\in[t_0,t_1]$$

$$\le \frac{1}{10b}\left[d + c\left(\sup|\, f(t,x(t),x(t-1),x(t-2),u(t),u(t-h))|\right) \; for \; t\in[t_0,t_1]\right]$$

$$|\,y(t)| \le a_3 + a_1\|v\|t_1 + t_1a_4\left(\sup|\, f(t,x(t),x(t-1),x(t-2),u(t),u(t-h)|\right) \; for \; t\in[t_0,t_1]$$

$$\le b\|v\| + \frac{d}{10} + \frac{c}{10}\left(\sup|\, f(t,x(t),x(t-1),x(t-2),u(t),u(t-h))|\,\right) for \; t\in[t_0,t_1]$$

Let *f* satisfy the following condition: for each pair of positive constant c and d, there exists a positive constant *r* such that, if $|p| \le r$, then $c|\, f(t,p)| + d \le r$ for all $t\in[t_0,t_1]$. Let *r* be chosen so that this implication is satisfied and $\sup_{-h\le t\le 0}|\phi(t)| \le \frac{r}{5}$. Therefore, if $\|x\| \le \frac{r}{5}$ and $\|u\| \le \frac{r}{5}$ then $|x(t)| + |x(t-1)| + |x(t-2)| + |u(t)| + |u(t-h)| \le r$ for all $t\in[t_0,t_1]$. It follows that $d + c\left(\sup|f(t,x(t),x(t-1),x(t-2),u(t),u(t-h))|\right)$ for $t\in[t_0,t_1] \le r$. Therefore, $|v(t)| \le \frac{r}{10b}$ for all $t\in[t_0,t_1]$ and hence $\|v\| \le \frac{r}{10b}$. It follows that $|y(t)| \le \frac{r}{10} +$
$\frac{r}{10}$ for all $t\in[t_0,t_1]$ and hence $\|y\| \le \frac{r}{5}$.

We have just shown that if *B*(*r*)=$\left\{(x,u)\in B : \|x\| \le \frac{r}{5} \; and \; \|u\| \le \frac{r}{5}\right\}$. Then *T*:*B*(*r*)→ *B*(*r*), where *B*(*r*) is a bounded set, hence *T* is well defined.

We shall now show that *T* is completely continuous or *T* is a sequentially compact operator. Since *f* is continuous, it follows that *T* is continuous. Let $B'(r)$ be a bounded subset of $B(r)$. Consider the sequence $\{(y_i, v_j)\} \in T(B')$, such that $(y_i, v_j) = T(x_j, u_j)$ for some $(x_j, u_j) \in B' \; for \; j = 1, 2, \ldots$. Since *f* is continuous, $|f(t,x(t),x(t-1),x(t-2),u(t),u(t\text{-}h))|$ is uniformly bounded for all $t\in[t_0,t_1]$. It follows that $\{y_i, v_j\}$ is a bounded sequence in $B(r)$. Hence $\{v_j(t)\}$ is an equicontinous and a uniformly bounded sequence on $[-h,t_1]$. Since each $v_j(t)$ has both right hand left hand limits at $t = t_0$ and $t = t_1 - h$, we can apply Ascoli's theorem on $[t_0, t_1 - h]$ to the sequence $\{v_j(t)\}$. Therefore, there exists a subsequence of $\{v_j(t)\}$ which converges uniformly to a continuous function on $[t_0, t_1 - h]$. Also, since $\{y_i(t)\}$ is a uniformly bounded and equicontinous sequence on $[-h,t_1]$, a further application of Ascoli's theorem yields a further subsequence $\{(y_i, v_j)\}$ which converges *B* to some (y_0, u_0). It follows that $T(B')$ is sequentially compact. Hence the closure $\{T(B')\}$ is sequentially compact. Thus, *T* is completely continuous. Since $B(r)$ is closed, bounded and convex, the Schauder fixed-point theorem implies that *T* has a fixed point $(x,u)\in B(r)$ which is the required solution of system (2.2) capable of satisfying the boundary conditions $x(t_0) = x_0$ and $x(t_1) = x_1$ for $t_1 > t_0$ and $x_0, x_1 \in E^n$.This by implication means that

$$x(t_1, \phi, f) = x_L(t_1) + \int_{t_0}^{t_1} Y(t, s)\, u(s)\, ds$$

$$+ \int_{t_0}^{t_1} X(t-s)\, f(s, x(s), x(s-1), x(s-2), u(s), u(s-h))\, ds$$

for $t \in [t_0, t_1]$ and $x(t) = \phi(t)$ for $t \in [-h, 0]$. Hence $x(t)$ is a solution of system (2.2) and

$$x(t_1, \phi, f) = x_L(t_1) + \int_{t_0}^{t_1} Y(t,s) Y^T(t,s) W^{-1}(t_1) \times$$

$$\left[x_1 - x_L(t_1) - \int_{t_0}^{t_1} X(t-s)\, f(s, x(s), x(s-1), x(s-2), u(s), u(s-h))\, ds \right] ds$$ Hence,

$$+ \int_{t_0}^{t_1} X(t-s)\, f(s, x(s), x(s-1), x(s-2), u(s), u(s-h))\, ds = x_1$$

system (2.2) is completely controllable on $[t_0, t_1]$.

***Corollary* 4.1**

In system (2.2) assume that

(i) (2.1) has $rank\ \hat{Q}_n(t_1) = n$

(ii) (2.1) is proper

(iii) *f* satisfies the condition $\lim_{|p| \to \infty} \frac{|f(t, p)|}{|p|} = 0$

Then system (2.2) is completely controllable on $[t_0, t_1]$.

Proof

Immediately from Proposition 3.1, 3.2 and Theorem 3.1

5.0 Example

Consider the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + C\, u(t) + c_0 \int_{-\infty}^{0} \exp(\nu\theta) x(t+\theta)\, d\theta \qquad (5.1)$$

and its perturbation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + C\, u(t) + c_0 \int_{-\infty}^{0} \exp(\nu\theta) x(t+\theta)\, d\theta$$

$$+ e^{-at} \sin\,(x(t) + x(t-1) + x(t-2)) \cos\,(u(t) + u(t-h)) \qquad (5.2)$$

where $A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

To see that system (5.1) is proper in any interval $[t_0, t_1]$, we use the determining equation for each s as follows

$$Q_0(0)=\begin{bmatrix}0\\1\end{bmatrix},\ \ Q_0(s)\equiv 0,\ s\neq 0$$
$$Q_1(0)=A_0C=\begin{bmatrix}1\\-2\end{bmatrix},\ \ Q(h)=A_1C=\begin{bmatrix}0\\-1\end{bmatrix}$$
$$Q_1(s)=0,\ s\neq 0, s\neq h$$
$$\hat{Q}_1(t)=\begin{bmatrix}0 & 1\\1 & -2\end{bmatrix},\ \ h>t>0$$
$$\hat{Q}_1(2h)=\begin{bmatrix}0 & 1 & 0\\1 & -2 & -1\end{bmatrix}=2$$

Since rank $\hat{Q}_2(t)=2$ for each $t_1>t_0$ the system (5.1) is relatively controllable on each interval $[t_0,t_1]$ on E^n. Moreover,

$$f(t,x(t),x(t-1),x(t-2),u(t),u(t-h))$$
$$=\left|e^{-at}\sin(x(t)+x(t-1)+x(t-2),)\cos(u(t)+u(t-h))\right|\leq e^{-at}\cdot 1$$

Hence, we conclude that system (4.2) by Theorem 3.1 is completely controllable on $[t_0,t_1]$

6.0 Conclusion

Sufficient conditions for the complete controllability of a nonlinear system with infinite delay are derived. These conditions were given by the relative controllability of the linear part and the application of Schauder's fixed point theorem on the perturbation function.

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