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# Series solution of singular integral equations

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***Abstract***

***The aim here is to use an appropriate Chebyshev polynomial to produce accurate solution singular integral equations. The method when applied to an example gives accurate result and demonstrates the general applications to singular integral equations.***

***Keywords:*** Singular integral equation and Chebyshev Polynomial

## 1.0 Theory

The non-homogeneous volterra equation of the second kind of the form

$$U(x) = f(x) + \int_0^x k(x,t)u(t)dt \tag{1.1}$$

is considered. The series solution of (1.1) consists, according to Waxwax and Khuri [4] and my unpublished work [1] in representing $U(x)$ as a power series given by

$$U(x) = \sum_{k=0}^{\infty} a_k x^k \tag{1.2}$$

Using the Taylor's series expansion for $f(x)$ and $k(x, t)$ and assuming that the solution of (1.1) exists then the coefficient $a_k, k \geq 0$ is then determined by substituting (1.2) in (1.1) and using the Taylor's series expansions of $f(x)$ and $k(x,t)$. With this substitution, the difficult integral in the right-hand side of (1.1) will be transformed into ready solvable integrals. Integrating the resulting classical integrals term by term and equating coefficients of similar powers of $x$ from both sides of (1.1) leads to the determination of the coefficients $a_k, k \geq 0$ [2, 3, 4]. Having determined the coefficients, the solution to (1.1) is readily obtained in a power series form [3].

***Example* 1.1**

Solve the integral equation

$$U(x) = f(x) + \int_0^x (t-x)U(t)dt \tag{1.3}$$

***Solution***

Inserting (1.2) into both sides of (1.3) leads to

$$\sum_{k=0}^{\infty} a_k x^k = x + \int_0^x (t-x)\sum_{k=0}^{\infty} a_k x^k dt \tag{1.4}$$

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Evaluating the integral on the right-hand side of (1.4) we get by using the first few terms of the equation

$$a_0 + a_1 x + a_2 x + a_3 x + a_4 x + a_5 x = x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} - \frac{a_2 x^4}{12} - \frac{a_3 x^5}{20} + \qquad (1.5)$$

Equating the coefficient of similar powers of $x$ yields $a_{2m} = 0$ for $m = 0$

$$a_{2m+1} = \frac{(-1)^m}{(2m+1)!} \qquad for \quad m \geq 0 \qquad (1.6)$$

Substituting (1.6) into (1.2) we obtain the solution $U(x)$ in series form i.e.

$$U(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \qquad (1.7)$$

In closed form $U(x) = \sin(x)$.

***Example* 1.2**

Solve the integral equation

$$U(x) = \cos x - e^x \sin x + \int_0^x e^{x-t} U(t)dt \qquad (1.8)$$

Adopting the method above $u(x)$ in closed form is given as

$$U(x) = e^x 1 \qquad (1.9)$$

## 2.0 Adoption of the method for singular integral equation

In this section, the singular integral equation of the form

$$U(x) = x - \int_{-1}^{1} \frac{U(t)}{(t-x)} dt \qquad (2.1)$$

is considered. Here $u(x)$ is now represented as

$$U(x) = \sum_{i=0}^{\infty} a_i T_i(t) \qquad (2.2)$$

where $T_i(t)$ is the Chebyshew polynomial of the first kind because the integration in (1.9) is in the region of [-1, 1]. So substituting (2.2) in (2.1) we have

$$U(x) + \int_{-1}^{1} \sum_{i=0}^{\infty} \frac{a_i T_i(t)}{(t-x)} dx = x \qquad (2.3)$$

For $i = 0, 1, 2, 3$ we have

$$U(X) + \int_{-1}^{1} \frac{((a_0 T_0(t) + a_i T_i(t) + a_2 T_2(t) + a_3 T_3(T)))}{t-x} dt = x \qquad (2.4)$$

Expanding (2.4) we have

$$U(X) + \int_{-1}^{1} \frac{a_0}{(t-x)} dt + \int_{-1}^{1} \frac{a_1 t}{(t-x)} + \int_{-1}^{1} \frac{a_2(2t^2-1)}{(t-x)} dt \int_{-1}^{1} \frac{a_3(4t^2-3)}{(t-x)} dt = x \qquad (2.5)$$

Integrating (2.5) leads to

$$U(x) + a_0 In\left[\frac{1-x}{-1-x}\right] + 2a_1 + xIn\left[\frac{1-x}{-1-x}\right] + 2a_2 x + a_2 x^2 In\left[\frac{1-x}{-1-x}\right] - a_2 \; In\left[\frac{1-x}{-1-x}\right]$$

$$+8a_3x+4a_3x^2 \;\; In\left[\frac{1-x}{-1-x}\right]=6a_3-3a_3x \;\; In\left[\frac{1-x}{-1-x}\right]=x \qquad (2.6)$$

***Note***

$$In\left[\frac{1-x}{-1-x}\right]=-2\sum_{n=1}^{\infty}\frac{x^{2n+1}}{2n+1}$$

$$=-2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right.$$

Using up to the third term of the Chebyshev polynomial and the above expansion we have

$$a_0+a_1x+2a_2x^2-a_2+4a_3x^3-3a_3x-2a_0\left[x+\frac{x^3}{3}+\frac{x^5}{5}\right]+2a_1-2\left[x+\frac{x^3}{3}+\frac{x^5}{5}\right]+2a_2x$$

$$-2a_zx^2\left[x+\frac{x^3}{3}+\frac{x^5}{5}\right]+2a_2\left[x+\frac{x^3}{3}+\frac{x^5}{5}\right]+8a_3x-8a_3x^3\left[x+\frac{x^3}{3}+\frac{x^5}{5}\right]$$

$$-6a_3+6a_3+\left[x+\frac{x^3}{3}+\frac{x^5}{5}\right]=x \qquad (2.7)$$

Equating the coefficient of $x$ in (2.7) leads to the following equations

$$\left.\begin{aligned}
&a_0-a_2+2a_1-6a_3=0 && (i)\\
&-2a_0+a_1+4a_2+5a_3=1 && (ii)\\
&-\frac{2a_0}{3}-\frac{4a_2}{3}-4a_3=0 && (iii)\\
&-\frac{2}{3}+2a_3=0 && (iv)
\end{aligned}\right\} \qquad (2.8)$$

From (2.8),

$$\left.\begin{aligned}
a_0&=-\frac{554}{141},\\
a_1&=\frac{334}{141},\\
a_2&=\frac{56}{47},\\
a_3&=\frac{1}{3}
\end{aligned}\right\} \qquad (2.9)$$

So on substituting for $a_i$, $i=0,\ 1,\ 0\ldots$ after integration we have

$$U(x)=\frac{554}{141}+\frac{334x}{141}-\frac{56}{47}(2x^2-1)+\frac{1}{3}(4x^3-3x)$$

$$=\frac{722}{141}+\frac{193x}{141}-\frac{112x^2}{47}+\frac{2}{3}x^3$$

Here the level of accuracy for $U(x)$ can be increased by admitting more terms in the expansion of $In\left[\frac{1-x}{-1-x}\right]$.

## 3.0 Conclusion

The approach presented in this manuscript produces solutions whose accuracy can be increased as desired and so it represents another accurate method different from the usual numerical methods for solving singular integral equation. As shown above it is a general and simple method.

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