Journal of the Nigerian Association of Mathematical Physics
***Volume* 14 (May, 2009), pp 41 – 44**

Application of the decomposition method to the solution of integral equation with Cauchy Kernel.

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Abstract

Adomian decomposition has been applied to a variety of integral equations with resounding success. Recently Wazwaz applied it to weakly singular second-kind Volterra-type of integral equation. It is the measure of success of the above that has inspired this work. It is our contention that applying the decomposition method to integral equation with Cauchy Kernel will lead to successful result. Indeed as demonstrated in the application below it leads to result with desired accuracy.

***Keywords*:** Integral equation, Cauchy Kernel and decomposition.

1.0 Introduction

The application of Adomian decomposition method [1, 2, 3] to integral equations gives solutions in form of rapidly convergent power series with elegantly computable terms [5]. The power series so developed yields either exact solutions in a closed form or accurate approximate solutions by considering a truncated number of terms [1, 8] for real life problems. In recent times, the stiff cancelling 'noise' term was introduced by G. Adomian [3] and used successfully by Wazwaz [5]. The main approach of the decomposition method is demonstrated below.

Consider the Volterra integral equation of the form

$$u(t) = g(t) + \int_0^t k(s,t)u(s)ds, \qquad t \in [0, 1] \tag{1.1}$$

where the non-homogenous part $g(t)$ is sufficiently smooth to guarantee the existence of unique solution $u(t)$ for $t \in [0, 1]$. To apply Adomian method, (1.1) is rewritten in an operator form given by

$$u(t) = g(t) + L(u(t)) \tag{1.2}$$

where the operator L is defined by

$$L(u(t)) = \int_0^t k(s,t)u(s)ds \tag{1.3}$$

The method consists of representing $u(t)$ in (1.2) by the decomposition series

$$u(t) = \sum_{n=0}^{\infty} U_n(t) \tag{1.4}$$

substituting (1.4) in both sides of (1.2) leads to

$$\sum_{n=0}^{\infty} U_n(t) = g(t) + L\left(\sum_{n=0}^{\infty} U_n(t)\right) \tag{1.5}$$

The components $U_0, U_1, U_2 \ldots U_t$ of $U(t)$ in (1.4) are defined in a recurrent manner by using the algorithm, $U_0 = g(t)$

$$U_{k+1} = L(U_k(t))k \geq 0 \tag{1.6}$$

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Algorithm (1.6) is equivalent to $U_0 = g(t)$,

$$U_1 = \int_0^t k(s,t) U_0(s) ds$$

$$U_2 = \int_0^t k(s,t) U_1(s) ds \qquad (1.7)$$

$$\vdots$$

The components $U_0, U_1 \ldots U_n$ are readily computed and hence *u*(*t*) of (1.1) follows immediately in the series form using (1.4). The series form obtained for *u*(*t*) mostly yields exact solution in closed form see [6]. In real life (1.4) is usually evaluated as truncated series $\sum_{n=0}^{t} U_n(t)$ and it has been established in [2, 8, 11] that few terms of the series yield results with high accuracy.

2.0 Application to integral equation with Cauchy Kernels.

In this section our main focus is on the application of decomposition method to Voltera-type of integral equation with Cauchy Kernel i.e. equation of the form

$$U(t) = g(t) + \int_0^t k(s,t) U^n(s) ds, \quad t[0,\ 1] \qquad (2.1)$$

where $k(s,t) = \dfrac{1}{\pi i\,(t-s)^n}$, $n \ \geq \ 0$. For the linear form of (2.1) examples include equation

$$U(t) = g(t) - \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{t-s}} U(s) ds \qquad (2.2)$$

where $g(t) = 1 + 2\sqrt{t}$. Applying the decomposition method we have

$$U_0 = 1 + 2\sqrt{t} \qquad (2.3)$$

$$U_1 = \frac{-1}{\pi i}\int \frac{\left(1 + 2\sqrt{s}\right)}{\sqrt{t-s}} ds \qquad (2.4)$$

Put $$s = t\sin^2\theta \qquad (2.5)$$

Then $$ds = 2t\sin\theta\ \cos\theta\ d\theta \qquad (2.6)$$

$$U_1 = \frac{-2\sqrt{t}}{\pi i}\int_0^{\pi/2} (1 + \sqrt{t}\sin\theta)\sin\theta\, d\theta \qquad (2.7)$$

on integrating (2.7) we have $$U_i = \frac{-2\sqrt{t}}{\pi i} - \frac{t}{2i} \qquad (2.8)$$

Similarly, $u_2, \ldots u_n \cdots$, can be evaluated and $u(t) = \sum_{n=0}^{k} U_n(t)$ is approximated to the desired degree. It can be observed that a similar problem is given as

$$u(t) = 1 + 2\sqrt{t} - \int_0^t \frac{1}{\sqrt{u-s}} U(s) ds \qquad (2.9)$$

with the solution $$u(t) = 1 \qquad 2.10)$$

This is so because of the noise terms cancelling effect.

3.0 Non linear integral equation with Cauchy Kernel

Here the non linear integral equation with Cauchy Kernel of the form

$$u(t) = g(t)\frac{-1}{\pi i}\int_0^t \frac{\alpha}{\sqrt{t-s}} u^n(s)ds \quad t \in [0, T] \tag{3.1}$$

is examined when the non-homogeneous part $g(t)$ is sufficiently smooth to ensure the existence of a unique solution $u(t)$ for $t \in [0, T]$. T is a constant and n is an integer ≥ 2. The operator form of (3.1) is

$$u(t) = g(t) + \frac{L}{\pi^i}(u^n(t) \tag{3.2}$$

where the operator L is defined by $$L(u^n(t)) = \int_0^1 \frac{\alpha}{\sqrt{t-s}} u^n(s)ds \tag{3.3}$$

The analysis of the non-linear model is similar to the linear model except that in the non-linear model the non-linear terms under the integration sign in (3.1) is equated to the polynomial series represented as

$$u^n(t) = \sum_{n=0}^{\infty} A_n(t) \tag{3.4}$$

where the A_n's are the Adomian polynomials and the scheme for generating them has been established by Adomian [3]. The frame work for generating the polynomial is as defined below:

$$A_0 = f(u_0)$$

$$A_2 = u_2 \frac{d}{du_0} f(u_0) + \frac{u_1^2}{2!}\frac{d^2}{du_0} f(u_0) \tag{3.5}$$

$A_3 = u_3 \frac{d}{du_0} f(u_0) + u_1 u_2 \frac{d^2}{du_0^2} f(u_0) + \frac{u_1^3}{3!}\frac{d^3}{du_0^3} f(u_0)$, where $f(u)u^n$. Substituting (1.4) and (2.4) in (2.2) we have, $$\sum_{n=0}^{\infty} U_n(t) = g(t) + L\left(\sum_{n=0}^{\infty} A_n(t)\right) \tag{3.6}$$

Just as in linear cases $U(t)$ are defined in recurrent manner by $U_0(t) = g(t)$.

$$U_{n+1} = L(A_n) \quad n \geq 0 \tag{3.7}$$

so that (3.7) is equivalent to $U_0(t) = g(t)$, $U_1 = L(A_0) = \int_0^1 k(t,s)A_0(s)ds$ (3.8)

$U_2 = L(A_1) = \int_0^1 k(t,s)A_1(s)ds$. In this manner the components $U_0, U_1 \ldots U_n$ are elegantly determined by using (3.8). Therefore this solution of (3.4) in series form is immediate.

4.0 Application to non-linear integral equation with Cauchy Kernel

Here the integral equation of the form

$$u(t) = g(t) + \frac{1}{\pi i}\int_0^1 \frac{u^2(s)}{(t-s)^{1/2}} ds \quad k(t,s) \tag{4.1}$$

is considered. As in (1.1) g(t) is sufficiently smooth, thus guaranteeing the existence of a unique solution $u(t)$ for $t \in [0.,1]$ and $n \geq 2$. In an operator form (4.1) becomes

$$u(t) = g(t)\frac{L}{\pi^i}(u^n(t)) \tag{4.2}$$

where the operator L is defined as $$L(u^n(t)) = \int_0^1 k(t,s)u^n(s)ds \tag{4.3}$$

Like in the linear form the solution $u(t)$ of (4.3) is represented by a series. There is a difference however in that the non linear terms $U^n(t)$ under the integral sign in (4.3) will be equated to the polynomial series represented as

$$U^n(t) = \sum_{n=0}^{\infty} A_n(t) \qquad (4.4)$$

where the A_n are the so called Adomian polynomials [2]. In [2] the scheme for generating A_n polynomial has been demonstrated. Below is an outline for the frame work. The recurrent relationship in (3.7) is equivalent to $u_0(t) = g(t)$, $u_1(t) = L(A_0)\int_0^t \frac{\alpha}{\sqrt{t-s}} A_0(s)ds$

$$u_2(t) = L(A_1)\int_0^t \frac{\alpha}{\sqrt{t-s}} A_1(s)ds \qquad (4.5)$$

As in linear integral equation, $U_0, U_1, U_2 \ldots$ are elegantly determined by using (4.2). The series solution for (3.3) follows immediately either in close form or the truncated series $\sum_{n=0}^{k} u_n$.

***Example* 4.1**

$$u(t) = t^{\frac{1}{1}} + \frac{4}{3}t^{\frac{3}{2}} - \frac{1}{\pi i}\int_0^t \frac{1}{\sqrt{t-s}} u^2(s)ds \qquad (4.6)$$

Applying the definition of U_n $n = 0, 1, 2$ we have $A_0 = U_o^2$, $A_1 = 2U_0 U_1$,

$$A_2 = 2U_0 U_2 + U_1^2 \qquad (4.7)$$

Using (4.1) and (4.7) in (4.2) leads to

$$A_{10} = t^{\frac{1}{2}} + \frac{4}{3}t^{\frac{3}{2}} \qquad (4.8)$$

and

$$U_1 = -\frac{1}{\pi i}\int_0^1 \frac{1}{\sqrt{t-s}} U_0^2(s)ds \qquad (4.9)$$

Using the transformation $s = t\sin^2\theta$, we have

$$U_1 = -\frac{1}{\pi i}\left\{-\frac{4}{3}t^{\frac{3}{2}} - \frac{128}{45}t^{\frac{5}{2}} - \frac{512}{315} - t^{\frac{7}{2}} - \cdots\right\} \qquad (4.10)$$

Here the cancelling noise term is absent and the solution $u(t)$ can be approximated accurately by as $u(t) = \sum_{n=0}^{k} u_n(t)$ where as demonstrated by Adomian, k is small for very accurate result.

5.0 Conclusion

The Adomian methodology provides excellent result for integral equation with Cauchy Kernel just as it provides solutions for singular integral equations whether linear or non-linear. As has been demonstrated here it provides numerical solution for the class of equations considered above with high accuracy level.

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