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On a lightly damped elastic quadratic model structure modulated by a dynamic periodic load

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Abstract

In this paper, we employ perturbation procedures in asymptotic expansions of the various variables to determine the dynamic buckling load of a lightly damped elastic quadratic model structure modulated by a dynamic periodic load. We finally relate the dynamic buckling load to its static equivalent and show that given any one of them, the other can automatically be obtained.

1.0 Introduction

The elastic stability of most engineering materials when loaded statically or dynamically, is a major concern to practitioners of engineering profession and Applied Mathematicians alike ,at least ,from the perspective of design and production. Most engineering structures fail when loaded beyond the stability limit and initial imperfections have been seriously implicated as major causes of such failures. The initial imperfection could be deterministic or stochastic, and, in most cases, are normally introduced inadvertently into the structure during the manufacturing process. Budiansky and Hutchinson [1-3] were the first to prop into the dynamic stability of such structures and thus pioneered early analyses of imperfection-sensitivity of these structures. Amongst the many early topics of their investigation, was the elastic quadratic model structure which is a mathematical generalization of most commonly used engineering structures. In this investigation, we shed further light on the understanding of the dynamic stability of a lightly and viscously damped quadratic structure modulated by a periodic load.

2.0 Formulation

A simple structure that ably captures the mathematical details of a quadratic structure is a two-arm simply-supported column, each of length *L*, carrying a mass *M* at their meeting point from where is attached a nonlinear spring of spring-characteristic *K* (see Figure 2.1).

The spring produces a nonlinear restoring force per unit length of value $\overline{F} = KL\left(\xi - \alpha\xi^2\right)$, where ξ is the additional movement of the column from the equilibrium position . We assume $\overline{\xi}$ to be the initial displacement. The equation of equilibrium as in [1-3] is

$$\frac{d^2\xi}{dt^2} + (1-\lambda)\xi - \alpha\xi^2 = \lambda\overline{\xi}\,\overline{F}(t)\ ;\quad \xi(0) = \frac{d\xi(0)}{dt} = 0\,,\ 0<\lambda<1,\ 0<\overline{\xi}<1 \qquad (2.1)$$

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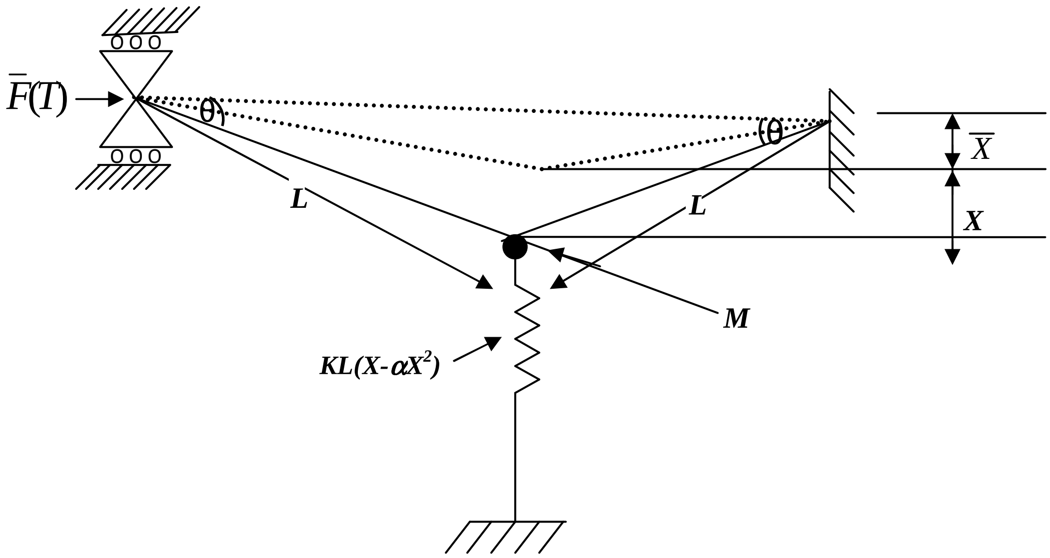

Figure 2.1: A simple Quadratic – Elastic Model

where α is the imperfection-sensitivity parameter and λ is a load parameter which has been nondimensionalized with respect to the classical buckling load λ_C given by $\lambda_C = \frac{KL}{2}$.Here $\bar{F}(t)$ is the loading history, which , in [1-3] ,was the step load given by $\bar{F}(t)$=1. The critical static buckling load λ_S is obtained by neglecting the inertia term, setting $\bar{F}(t)$=1 and using the condition $\frac{d\lambda}{d\xi} = 0$ to obtain

$$(1-\lambda_S)^2 = 4(\alpha\bar{\xi})\lambda_S \tag{2.2}$$

For the step loading case, we set $\bar{F}(t)$=1 and use a similar process to obtain the dynamic buckling load λ_D as

$$(1-\lambda_D)^2 = \frac{16}{3}(\alpha\bar{\xi})\lambda_D \tag{2.3}$$

On eliminating $\alpha\bar{\xi}$ from (2.2) and (2.3), we have

$$\frac{\lambda_D}{\lambda_S} = \frac{3}{4}\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2 \tag{2.4}$$

Danielson [4] made a two-fold refinement of the Budiansky/Hutchinson model by introducing an additional spring with spring K_0 and mass M_0 (see Figure 2.2) with the aim of stimulating pre-buckling motion, and so derived the following coupled equations, which we have further improved in this investigation by introducing uniform viscous damping on the two modes:

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2} + c_0\frac{d\xi_0}{dT} + \xi_0 + \frac{K_0\xi_1(\xi_1 + 2\bar{\xi})}{\lambda_C} = \lambda\bar{F}(T) \tag{2.5}$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2} + c_0\frac{d\xi_1}{dT} + \xi_1(1-\xi_0) - \alpha\xi_1^2 + \frac{K_0}{\lambda_C}\xi_1(\xi_1 + \bar{\xi})(\xi_1 + 2\bar{\xi}) = \bar{\xi}\xi_0 \tag{2.6}$$

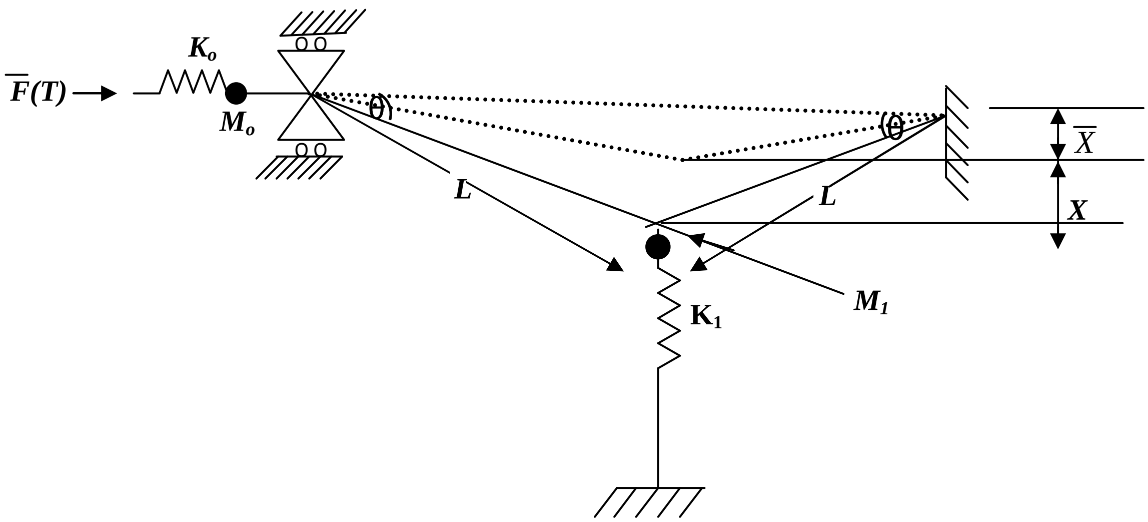

Figure 2.2: A simple Quadratic – Elastic Model Structure

$$\xi_i(0)=\frac{d\xi_i(0)}{dT}=0\,;\,\mathrm{i}=0{,}1\,;\,\lambda_{\mathrm{C}}=\frac{K_1}{2},\,\omega_0=\left(\frac{K_0}{m_0}\right)^{\frac{1}{2}},\,\omega_1=\left(\frac{K_1}{m_1}\right)^{\frac{1}{2}} \tag{2.7}$$

where ω_0 and ω_1 are the circular frequencies of the pre-buckling mode $\xi_0(T)$ and buckling mode $\xi_1(T)$ respectively and T is the time variable. Danielson, using a perturbation analysis in a Mathieu-type of instability, solved the undamped version of equations (2.5) and (2.6) for the case $\overline{F}(T)=1$ (i.e. step load), neglecting the pre-buckling inertia, and obtained the following results in two intervals of ratio of frequencies, namely

$$\left(\frac{\lambda_D}{\lambda_S}\right)=\frac{3}{4}\left(\frac{1-\lambda_{\mathrm{D}}}{1-\lambda_{\mathrm{S}}}\right)^2,\text{for } 0<\left(\frac{\omega_1}{\omega_0}\right)<\frac{1}{2} \tag{2.8}$$

$$\left(\frac{\lambda_D}{\lambda_S}\right)=\frac{\frac{1}{6}\left\{4-\left(\frac{\omega_0}{\omega_1}\right)^2\right\}}{\lambda_{\mathrm{S}}+\frac{10}{9}\left(\frac{\omega_1}{\omega_0}\right)^2\left(1-\lambda_{\mathrm{S}}\right)^2},\ \text{for }\ \frac{1}{2}<\left(\frac{\omega_1}{\omega_0}\right)<1 \tag{2.9}$$

In the present study, we extend Danielson's analysis by making three major refinements thus: (a) the inclusion of a dynamic periodic load as was the case in [5] ,(b) the introduction of light viscous damping, and (c) the non-neglect of the pre-buckling inertia term .Unlike Danielson's study, we avoid the use of Mathieu-type of instability, for as noted by Budiansky [3,page 100] , this type of instability is usually associated with many cycles of oscillation as opposed to just one shot of oscillation that normally triggers off dynamic buckling .Analyses similar to the present one were done by Svalbonas [6] , Wei et al [7] ,Batra and Wei [8] and Zhang et al [9] among others .

3.0 Method of solution

We let $t=\omega_0 T,\ \mathrm{f}(\mathrm{t})=\overline{F}\left(\frac{t}{\omega_0}\right)=\cos\theta\,\mathrm{t}$, and note that $\frac{d\xi_i}{dT}=\omega_0\frac{d\xi_i}{dt}$. Thus , the governing equations (2.5) -(2.7) become

$$\frac{d^2\xi_0}{dt^2}+2\bar{\xi}\delta_0\frac{d\xi_0}{dt}+\xi_0-\frac{K_0}{\lambda_C}\xi_1\left(\xi_1+2\bar{\xi}\right)=\in P\cos\theta\,\mathrm{t} \quad (3.1)$$

$$\frac{d^2\xi_1}{dt^2}+2\bar{\xi}\delta_1\frac{d\xi_1}{dt}+Q^2\left(\xi_1-\xi_1\xi_0\right)-\alpha Q^2\xi_1^2+\frac{K_0}{\lambda_\mathrm{C}}Q^2\left(\xi_1^3+2\bar{\xi}\xi_1^2+2\bar{\xi}^2\xi_1\right)=Q^2\bar{\xi}\xi_0 \quad (3.2)$$

$$\xi_j(0)=\frac{d\xi_j(0)}{dt}=0\ ,\ j=0,2 \quad (3.3)$$

where

$$2\delta_0\bar{\xi}=c_0\omega_0\ ,\ 2\delta_1\bar{\xi}=c_0\left(\frac{\omega_1^2}{\omega_0}\right),0<\delta_0\ ,\delta_1<1,\ Q=\frac{\omega_1}{\omega_0},0<\frac{\omega_1}{\omega_0}<1,\mathrm{P}=\frac{1}{\mathrm{Q}^2},\in=\lambda\,\mathrm{Q}^2\ ,0<\in<1$$

We consider $0<\bar{\xi}<1$ and assume that both $\in$ and $\bar{\xi}$ are mathematically unrelated. These two parameters are considered small compared to unity .In our quest for solution, we are to determine a particular value of λ, namely λ_D,called the dynamic buckling load , for which the structure becomes dynamically unstable. We define λ_D as the largest load parameter for which the solution of equations (3.1)-(3.3) remains bounded for all time $t>0$. We shall [1-5] determine λ_D from the maximization

$$\frac{d\lambda}{d\xi_m}=0\,,\,\xi_\mathrm{m}=\xi_{0\max}+\xi_{1\max} \quad (3.4)$$

where $\xi_{0\max}$ and $\xi_{1\max}$ are the maximum values of $\xi_0(t)$ and $\xi_1(t)$ respectively. Our initial pre-occupation will now be to determine $\xi_{0\max}$ and $\xi_{1\max}$ which we henceforth embark. Letting $\tau=\bar{\xi}\ t$, we have

$$\frac{d\xi_i}{dt}=\xi_{i,\mathrm{t}}+\bar{\xi}\xi_{i,\tau}\ ,\frac{\mathrm{d}^2\xi_i}{dt^2}=\xi_{i,\mathrm{t\,t}}+2\bar{\xi}\xi_{i,\mathrm{t}\tau}+\bar{\xi}^2\xi_{i,\tau\tau}\ ,\mathrm{i}=0,1.$$

where a subscript following a comma indicates partial differentiation. We let

$$\xi_0(t)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\zeta^{i\,\mathrm{j}}(t,\tau)\in^i\ \bar{\xi}^j\ \ ,\ \ \xi_1(t)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\eta^{i\,\mathrm{j}}(t,\tau)\in^i\ \bar{\xi}^j \quad (3.5)$$

where the terms ij ,as in $\zeta^{i\,\mathrm{j}}$ and $\eta^{i\,\mathrm{j}}$,are superscripts and not powers. On substituting into (3.1)-(3.3), using (3.5), we get the following:

$$L^{(1)}\zeta^{10}\equiv\zeta^{10}_{,\mathrm{t\,t}}+\zeta^{10}=P\cos\theta t \quad (3.6)$$

$$L^{(1)}\zeta^{11}=-2\zeta^{10}_{,\mathrm{t}\tau}-2\delta_0\zeta^{10}_{,\mathrm{t}} \quad (3.7)$$

$$L^{(1)}\zeta^{12}=-2\zeta^{11}_{,\mathrm{t}\tau}-2\delta_0\zeta^{11}_{,\mathrm{t}}-\zeta^{10}_{,\tau\tau}-2\delta_0\zeta^{10}_{,\tau} \quad (3.8)$$

$$L^{(1)}\zeta^{20}=0 \quad (3.9)$$

$$L^{(1)}\zeta^{21}=-2\zeta^{20}_{,\mathrm{t}\tau}-2\delta_0\zeta^{20}_{,\mathrm{t}} \quad (3.10)$$

$$L^{(1)}\zeta^{22}=-2\zeta^{21}_{,\mathrm{t}\tau}-2\delta_0\zeta^{21}_{,\mathrm{t}}-\zeta^{20}_{,\tau\tau}-2\delta_0\zeta^{20}_{,\tau}+\frac{K_0}{\lambda_\mathrm{C}}\left\{\left(\eta^{11}\right)^2+2\eta^{11}\right\} \quad (3.11)$$

$$L^{(2)}\eta^{11}\equiv\eta^{11}_{,t\,\mathrm{t}}+Q^2\eta^{11}=Q^2\zeta^{10} \quad (3.12)$$

$$L^{(2)}\eta^{12}=Q^2\zeta^{11}-2\eta^{11}_{,t\tau}-2\delta_1\eta^{11}_{,t} \quad (3.13)$$

$$L^{(2)}\eta^{21}=Q^2\eta^{11}\zeta^{10} \qquad (3.14)$$

$$L^{(2)}\eta^{22}=Q^2\left(\eta^{11}\zeta^{11}+\eta^{12}\zeta^{10}\right)+\alpha Q^2\left(\eta^{11}\right)^2-2\eta^{21}_{,t\tau}-2\delta_1\eta^{21}_{,t} \qquad (3.15)$$

The initial conditions are evaluated at $(t,\tau)=(0,0)$ as

$$\zeta^{ij}=0, \zeta^{10}_{,t}=0, \zeta^{1k}_{,t}+\zeta^{1p}_{,\tau}=0, \zeta^{20}_{,t}=0, \zeta^{2k}_{,t}+\zeta^{2p}_{,\tau}=0,\ i=1,2,3,\cdots; j=0,1,2,3,\cdots \qquad (3.16a)$$

$$\eta^{ij}=0, \eta^{10}_{,t}=0, \eta^{1k}_{,t}+\eta^{1p}_{,\tau}=0, \eta^{20}_{,t}=0, \eta^{2k}_{,t}+\eta^{2p}_{,\tau}=0, \mathrm{p}=\mathrm{k}-1,\ \mathrm{k}=1,2,3,\cdots;$$
$$i=1,2,3,\cdots; j=1,2,3,\cdots \qquad (3.16b)$$

On solving (3.6a) with appropriate initial conditions in (3.16a), we have

$$\zeta^{10}=\alpha_{10}(\tau)\cos t+\beta_{10}(\tau)\sin t+\tilde{P}\cos\theta\,\mathrm{t}\ ,\ \tilde{\mathrm{P}}=\frac{P}{1-\theta^2}, \theta\neq 1 \qquad (3.17a)$$

$$\alpha_{10}(0)=-\tilde{P}\ ,\ \beta_{10}(0)=0 \qquad (3.17b)$$

We now substitute for ζ^{10} in (3.7), and to ensure a uniformly valid solution in the time scale t, equate to zero the coefficients of cost and *sint* to get

$$\beta'_{10}+\delta_0\beta_{10}=0 \quad \text{and} \quad \alpha'_{10}+\delta_0\alpha_{10}=0 \qquad (3.18a)$$

where $(\ )'=\dfrac{d(\ )}{d\tau}$. We solve (3.18a) together with (3.17b) to get

$$\alpha_{10}(\tau)=\alpha_{10}(0)e^{-\delta_0\tau}, \beta_{10}(\tau)=0;\ \alpha'_{10}(0)=-\delta_0\alpha_{10}(0)=\delta_0\tilde{P} \qquad (3.18b)$$

Thus we have

$$\zeta^{10}=\alpha_{10}(\tau)\cos t+\tilde{P}\cos\theta\,\mathrm{t} \qquad (3.19)$$

On solving the remaining equation in (3.7), we have

$$\zeta^{11}(t,\tau)=\alpha_{11}(\tau)\cos t+\beta_{11}(\tau)\sin t, \alpha_{11}(0)=0, \beta_{11}(0)=-\alpha'_{10}(0)=-\delta_0\tilde{P} \qquad (3.20)$$

We next substitute the relevant terms into (3.8) and ,to ensure a uniformly valid solution in the time scale t, equate to zero the coefficients of cost and sint to get respectively

$$\beta'_{11}+\delta_0\beta_{11}=\frac{1}{2}\left(\alpha''_{10}+2\delta_0\alpha'_{10}\right) \text{ and } ,\alpha'_{11}+\delta_0\alpha_{11}=0 \qquad (3.21)$$

On solving (3.21), we get

$$\beta_{11}(\tau)=\frac{1}{2}e^{-\delta_0\tau}\left[\int_0^\tau\left(\alpha''_{10}(s)+2\delta_0\alpha'_{10}(s)\ \right)e^{\delta_0\tau}ds+2\beta_{11}(0)\right],\ \alpha_{11}(\tau)=0 \qquad (3.22)$$

The remaining equation in (3.8) is solved to get

$$\zeta^{12}=\alpha_{12}(\tau)\cos t+\beta_{12}(\tau)\sin t, \alpha_{12}(0)=0\ ,\ \beta_{12}(0)=0 \qquad (3.23)$$

We now substitute for ζ^{10} from (3.19) into (3.12) and get

$$L^{(2)}\eta^{11}=\eta^{11}_{,tt}+Q^2\eta^{11}=Q^2\left(\alpha_{10}\cos t+\tilde{P}\cos\theta\,\mathrm{t}\right) \qquad (3.24)$$

The solution of (3.24) using appropriate initial conditions is

$$\eta^{11}=a_{11}(\tau)\cos Q\,t+b_{11}(\tau)\sin Q\,\mathrm{t}+\mathrm{Q}^2\left(\frac{\alpha_{10}\cos t}{Q-1}+\frac{\tilde{P}}{\left(Q^2-\theta^2\right)}\right) \qquad (3.25a)$$

$$a_{11}(0)=-Q^2\tilde{P}\,r_6, r_6=\left(\frac{1}{Q^2-\theta^2}-\frac{1}{Q^2-1}\right), b_{11}(0)=0 \qquad (3.25b)$$

We next substitute into (3.13) and to ensure uniformly valid solution in *t*, equate to zero the coefficients of cos*Qt* and sin*Qt* to get respectively

$$b'_{11}+\delta_1 b_{11}=0 \quad \text{and} \quad a'_{11}+\delta_1 a_{11}=0 \qquad (3.26a)$$

On solving (3.26a), we get

$$b_{11}=0\ ,\ a_{11}(\tau)=a_{11}(0)e^{-\delta_1\tau}\ ,\ a'_{11}(0)=-\delta_1 a_{11}(0) \qquad (3.26b)$$

Therefore we have

$$\eta^{11}=a_{11}(\tau)\cos Q\,t+Q^2\left(\frac{\alpha_{10}\cos t}{Q^2-1}+\frac{\tilde{P}}{\left(Q^2-\theta^2\right)}\right) \qquad (3.26c)$$

The remaining equation in (3.13) is solved to get

$$\eta^{12}=a_{12}\cos Q\,t+b_{12}\sin Q\,t+\frac{H_1\sin t}{Q^2-1}+\frac{H_2\sin\theta\,t}{Q^2-\theta^2} \qquad (3.27a)$$

where

$$H_1(\tau)=\left[Q^2\beta_{11}-\frac{2Q^2\alpha'_{10}}{Q^2-1}-\frac{2\delta_1 Q^2\alpha_{10}}{Q^2-1}\right],\ H_2(\tau)=\frac{2\delta_1 Q^2\tilde{P}\theta}{(Q^2-1)},\ H_1(0)=-\tilde{P}Q^2 h_1 \qquad (3.27b)$$

$$h_1=\left\{\delta_0+\frac{2\delta_0\delta_1}{Q^2-1}-\frac{2}{Q^2-1}\right\},\ H_2(0)=\tilde{P}Q^2 h_2\ ,\ h_2=\frac{2\delta_1\theta}{(Q^2-\theta)} \qquad (3.27c)$$

Thus we have

$$a_{12}(0)=0\ ,\ b_{12}(0)=\tilde{P}Qh_3\ ,\ h_3=\left\{\frac{h_1}{Q^2-1}-\frac{\theta\ h_2}{Q^2-\theta^2}-\delta_1 r_6-\frac{\delta_0}{Q^2-1}\right\} \qquad (3.27d)$$

The solution of (3.9) is

$$\zeta^{20}=\alpha_{20}\cos t+\beta_{20}\sin t\ ,\alpha_{20}(0)=\beta_{20}(0)=0 \qquad (3.28)$$

We substitute into (3.10) and maintain a uniformly valid solution in the time scale *t* to get

$$\alpha_{20}(\tau)=\beta_{20}(\tau)=0 \quad \textit{and} \quad \zeta^{20}(t,\tau)=0 \qquad (3.29)$$

The solution of the remaining equation in (3.10) is

$$\zeta^{21}=\alpha_{21}\cos t+\beta_{21}\sin t\ ,\alpha_{21}(0)=\beta_{21}(0)=0 \qquad (3.30)$$

We next substitute into (3.11) and equate to zero the coefficients of cost and *sint* and obtain respectively

$$\beta'_{21}+\delta_0\beta_{21}=-\frac{K_0 Q^2\alpha_{10}}{\lambda_C\left(Q^2-1\right)}\ ,\ \text{and} \quad \alpha'_{21}+\delta_0\alpha_{21}=0 \qquad (3.31a)$$

We solve (3.31a) to get

$$\beta_{21}(\tau)=-\frac{K_0 Q^2 e^{-\delta_0\tau}}{\lambda_C\left(Q^2-1\right)}\int_0^\tau \alpha_{10}(s)\,e^{\delta_0 s}\,ds\ ,\ \alpha_{21}(\tau)=0,\quad \beta'_{21}(0)=\frac{K_0 Q^2\tilde{P}}{\lambda_C\left(Q^2-1\right)} \qquad (3.31b)$$

The remaining equation in (3.11) is now

$$L^{(1)}\zeta^{22}=\frac{K_0}{\lambda_C}\Big[R_{23}+R_{24}\cos 2Q\,t+R_{25}\{\cos(1+Q)\,t+\cos(1-Q)\,t\}+R_{26}\{\cos(\theta+Q)\,t+\cos(Q-\theta)\,t\}$$

$$+R_{27}\cos 2t+R_{28}\cos 2\theta\,t+R_{29}\{\cos(1+\theta)\,t+\cos(1-\theta)\,t\}+2Q^2\left\{\frac{\alpha_{10}\cos t}{Q^2-1}+\frac{\tilde{P}\cos\theta\,t}{\left(Q^2-\theta^2\right)}\right\}\Big] \qquad (3.32)$$

$$\zeta^{22}(0,0)=0\,,\ \zeta^{22}_{,t}(0,0)+\zeta^{21}_{,\tau}(0,0)=0 \quad (3.33)$$

where

$$R_{23}=\frac{a_{11}^2}{2}+\frac{Q^4\alpha_{10}^2}{2(Q^2-1)}+\frac{\tilde{P}^2}{2(Q^2-\theta^2)},\ \mathrm{R}_{23}(0)=(\tilde{P}Q^2)^2 r_{23}\,, \quad (3.34)$$

$$\mathrm{r}_{23}=\frac{1}{2}\left\{r_6^2+\frac{1}{Q^2-1}+\frac{1}{Q^4(Q^2-\theta^2)}\right\},R_{24}(\tau)=\frac{a_{11}^2}{2},R_{24}(0)=(\tilde{P}Q^2)^2r_{24}\ ,r_{24}=\frac{r_6^2}{2} \quad (3.35)$$

$$R_{25}(\tau)=Q^2a_{11}\alpha_{10},\qquad R_{25}(0)=(\tilde{P}Q^2)^2r_6\,,\ \ \mathrm{R}_{26}(\tau)=\frac{Q^2\tilde{P}a_{11}}{(Q^2-\theta^2)},R_{26}(0)=(\tilde{P}Q^2)^2r_{26} \quad (3.36)$$

$$r_{26}=\quad -\mathrm{r}_6\,,\mathrm{R}_{27}(\tau)=\frac{Q^4\alpha_{10}^2}{2(Q^2-1)},R_{27}(0)=(\tilde{P}Q^2)^2r_{27}\,,\mathrm{r}_{27}=\frac{1}{2(Q^2-1)} \quad (3.37)$$

The

$$R_{28}(\tau)=\frac{\tilde{P}^2}{2(Q^2-\theta^2)},R_{28}(0)=(\tilde{P}Q^2)^2r_{28}\ ,\mathrm{r}_{28}=\frac{1}{2(Q^2-\theta^2)} \quad (3.38)$$

$$R_{29}(\tau)=\frac{\alpha_{10}\tilde{P}}{2(Q^2-1)(Q^2-\theta^2)},R_{29}(0)=\tilde{P}^2r_{29}\ ,r_{29}=-\frac{1}{2(Q^2-1)(Q^2-\theta^2)} \quad (3.39)$$

solution of (3.32a,b) is

$$\alpha_{22}(0)=-\frac{K_0}{\lambda_{\mathrm{C}}}\left(\tilde{P}Q^2\right)^2r_{30},\mathrm{r}_{30}=\left[r_{23}+\frac{r_{24}}{1-4Q^2}+\frac{r_6}{\mathrm{Q}}\left(\frac{1}{2-Q}-\frac{1}{2+Q}\right)\right.$$

$$+r_{26}\left\{\frac{1}{1-(Q^2-\theta^2)}+\frac{1}{1-(Q^2+\theta^2)}\right\}-\frac{2r_{26}}{Q^2\tilde{P}(1-Q^2)}-\frac{r_{27}}{3}+\frac{r_{28}}{Q^4(1-4\theta^2)} \quad (3.40)$$

$$\left.+\frac{r_{29}}{\theta(Q^3\tilde{P})}\left\{\frac{1}{2-\theta}-\frac{1}{2+\theta}\right\}+\frac{2}{Q^2\ \tilde{P}(Q^2-\theta^2)}\right],\ \beta_{22}(0)=0$$

We next substitute into (3.14) and get

$$L^{(2)}\eta^{21}=R_0+R_1\cos 2t+R_2\cos 2\theta\,\mathrm{t}+\mathrm{R}_3\cos(\theta+1)\,\mathrm{t}+\mathrm{R}_4\cos(1-\theta)\,\mathrm{t}$$

$$+\mathrm{R}_5\{\cos(Q+\theta)\,\mathrm{t}+\cos(\theta\text{-}\mathrm{Q})\,\mathrm{t}\}++\mathrm{R}_6\{\cos(Q+1)\,\mathrm{t}+\cos(1\text{-}\mathrm{Q})\,\mathrm{t}\}] \quad (3.41a)$$

$$\eta^{21}(0,0)=0\ \ ,\eta^{21}_{,t}(0,0)=0 \quad (3.41b)$$

$$R_0=\frac{Q^2}{2}\left\{\frac{\tilde{P}^2}{(Q^2-\theta)}+\frac{\alpha_{10}^2Q^2}{Q^2-1}\right\},R_0(0)=(\tilde{P}Q)^2r_{0,}\ \ r_0=\frac{1}{2}\left(\frac{1}{Q^2-\theta^2}+\frac{1}{Q^2-1}\right) \quad (3.41c)$$

$$R_1=\frac{\alpha_{10}^2Q^2}{2(Q^2-1)},\mathrm{R}_1(0)=(\tilde{P}Q)^2r_1\,,\mathrm{r}_1=\frac{1}{2(Q^2-\theta^2)}\quad,\mathrm{R}_2=\frac{\tilde{P}^2Q^2}{2\,(Q^2-\theta^2)}, \quad (4.41\mathrm{d})$$

$$\mathrm{R}_2(0)=(\tilde{P}Q)^2r_2\quad,r_2=\frac{1}{2(Q^2-\theta^2)} \quad (3.41\mathrm{e})$$

$$\mathrm{R}_3=\frac{\tilde{P}\alpha_{10}Q^4}{2}\left\{\frac{1}{(Q^2-\theta)}+\frac{1}{(Q^2-1)}\right\},\mathrm{R}_3(0)=\left(\tilde{P}Q^2\right)^2r_3 \quad (3.41f)$$

$$r_3=\frac{1}{2}\left\{\frac{1}{(Q^2-\theta)}+\frac{1}{(Q^2-1)}\right\};\mathrm{R}_4=\frac{\tilde{P}\alpha_{10}Q^4}{2}\left\{\frac{1}{(Q^2-\theta)}+\frac{1}{(Q^2-1)}\right\},\mathrm{R}_4(0)=-\left(\tilde{P}Q^2\right)^2r_4 \quad (3.41g)$$

$$(r_3 = r_4), \mathrm{R}_5 = \frac{\tilde{P} a_{11} Q^4}{2}, \mathrm{R}_5(0) = -\left(\tilde{P} Q^2\right)^2 r_5, \mathrm{r}_5 = r_3, \mathrm{R}_6 = \frac{\alpha_{10} a_{11} Q^2}{2}, \mathrm{R}_6(0) = \left(\tilde{P} Q^2\right)^2 r_6, \qquad (3.41h)$$

$$r_6 = r_5 \qquad (3.41i)$$

The solution of (3.41a,b) is

$$\eta^{21}(t,\tau) = \Bigg[a_{21}(\tau)\cos Q\,\mathrm{t} + \mathrm{b}_{21}(\tau)\sin Q\,\mathrm{t} + \frac{\mathrm{R}_0}{\mathrm{Q}^2} + \frac{R_1 \cos 2t}{Q^2 - 4} + \frac{R_2 \cos 2\theta\, t}{Q^2 - 4\theta^2} + \frac{R_3 \cos(\theta+1)t}{Q^2 - (\theta+1)^2}$$

$$+ \frac{R_4 \cos(1-\theta)t}{Q^2 - (1-\theta)^2} + \frac{R_5}{\theta}\left\{ \frac{\cos(\theta - Q)t}{2Q - \theta} - \frac{\cos(\theta + Q)t}{2Q + \theta} \right\} + R_6 \left\{ \frac{\cos(1-Q)t}{2Q-1} - \frac{\cos(1+Q)t}{2Q+1} \right\} \Bigg] \qquad (3.42a)$$

We now

$$a_{21}(0) = -\left(\tilde{P} Q^2\right)^2 r_{77}, r_{77} = \Bigg[\frac{r_0}{Q^4} + \frac{r_1}{Q^2 - 4} + \frac{r_2}{Q^2 - 4\theta^2} + \frac{r_3}{Q^2 - (\theta+1)^2} - \frac{r_4}{Q^2 - (1-\theta)^2}$$

$$- \frac{r_5}{\theta}\left\{ \frac{1}{2Q^2 - 1} - \frac{1}{2Q^2 + 1} \right\} \Bigg], \ \mathrm{b}_{21}(0) = 0 \qquad (3.42b)$$

simplify the following multiplications needed in the next round of substitution into (3.15) as:

$$\eta^{11}\zeta^{11} = \frac{\beta_{11}}{2}\Big[a_{11}\{\sin(1+Q)\,\mathrm{t} + \sin(1-Q)\,\mathrm{t}\}$$

$$+ Q^2 \left\{ \left\{ \frac{\alpha_{10}(1 + 2\cos 2t)}{Q^2 - 1} + \frac{\tilde{P}}{(Q^2 - \theta^2)}\{\sin(1+Q)\,\mathrm{t} + \sin(1-Q)\,\mathrm{t}\} \right\} \right\} \Big] \qquad (3.43a)$$

$$\eta^{12}\zeta^{10} = \frac{\alpha_{10} b_{12}}{2}\{\sin(1+Q)\,\mathrm{t} + \sin(1-Q)\,\mathrm{t}\} + \frac{\tilde{P} b_{12}}{2}\{\sin(\theta+Q)\,\mathrm{t} + \sin(Q-\theta)\,\mathrm{t}\}$$

$$+ \frac{H_1 \alpha_{10} \sin 2t}{2(Q^2 - 1)} + \frac{H_1 \tilde{P}}{2(Q^2 - 1)}\{\sin(1+\theta)\,\mathrm{t} + \sin(1-\theta)\,\mathrm{t}\} + \frac{H_2 \alpha_{10}}{2(Q^2 - \theta)}\{\sin(1+\theta)\,\mathrm{t} + \sin(\theta - 1)\,\mathrm{t}\}$$

$$+ \frac{H_2 \tilde{P} \sin\theta\,\mathrm{t}}{2(Q^2 - \theta)} \Bigg] \qquad (3.43b)$$

If we now substitute into (3.15) and to ensure uniformly valid solution in *t*, equate to zero the coefficients of *cosQt* and *sinQt*, we get the following respective equations

$$b'_{21} + \delta_1 b_{21} = 0 \quad \text{and} \quad \mathrm{a}'_{21} + \delta_1 a_{21} = 0 \qquad (3.44a)$$

The solutions of (3.44a) are

$$b_{21}(\tau) = 0, \ \mathrm{a}_{21}(\tau) = a_{321}(0) e^{-\delta_1 \tau} \qquad (3.44b)$$

If we regroup the remaining terms in (3.15), we get

$$L^{(2)}\eta^{22} = R_7 + R_8 \sin 2t + R_9 \cos 2t + R_{10} \sin 2\theta\,\mathrm{t} + \mathrm{R}_{11}\cos 2\theta\,\mathrm{t} + \mathrm{R}_{12}\sin(1+\theta)\,\mathrm{t} + R_{13}\sin(\theta - 1)\,\mathrm{t}$$

$$+ \mathrm{R}_{14}\cos(1+\theta)\,\mathrm{t} + \mathrm{R}_{15}\cos(1-\theta)\,\mathrm{t} + \mathrm{R}_{16}\sin(Q+\theta)\,\mathrm{t} + \mathrm{R}_{17}\sin(Q-\theta)\,\mathrm{t} + \mathrm{R}_{18}\sin(Q+1)\,\mathrm{t}$$

$$+ \mathrm{R}_{19}\sin(1-Q)\,\mathrm{t} + \mathrm{R}_{20}\cos(1+Q)\,\mathrm{t} + \mathrm{R}_{21}\cos(1-Q)\,\mathrm{t} + R_{22}\{\cos(Q+\theta)\,\mathrm{t} + \cos(\mathrm{Q} - \theta)\,\mathrm{t}\} \qquad (3.45a)$$

$$\eta^{22}(0,0) = 0, \eta^{22}_{,\mathrm{t}}(0,0) + \eta^{21}_{,\tau}(0,0) \qquad (3.45b)$$

$$R_7 = \left[\frac{Q^4 \beta_{11} \alpha_{10}}{2(Q^2 - 1)} + \frac{\alpha Q^2}{2}\left\{ a_{11}^2 + Q^2 \alpha_{10}^2 + \frac{\tilde{P}^2}{(Q^2 - \theta^2)} \right\} \right] \qquad (3.46a)$$

$$R_8 = \left[\frac{Q^2 H_1 \alpha_{10}}{2(Q^2-1)} + \frac{4\delta_1 R_1'}{Q^2-4} + \frac{4\delta_1 R_1}{Q^2-4}\right], \mathrm{R}_9 = \frac{1}{2}\left[\frac{Q^4 \beta_{11}\alpha_{10}}{(Q^2-1)} + \frac{\alpha\, Q^8 a_{11}\alpha_{10}^2}{(Q^2-1)}\right] \qquad (3.46b)$$

$$R_{10} = \left[\frac{Q^2 H_2 \tilde{P}}{2(Q^2-\theta^2)} + \frac{4\delta_1 \theta\, R_2}{(Q^2-4\theta^2)}\right], \mathrm{R}_{11} = \left[\alpha Q^2 a_{11}^2 \frac{\alpha\, Q^4 \tilde{P}^2 a_{11}}{(Q^2-\theta^2)}\right] \qquad (3.46c)$$

$$R_{12} = \frac{1}{2}\left[\frac{Q^2 H_1 \tilde{P}}{(Q^2-1)} + \frac{H_2 \alpha_{10} Q^2}{(Q^2-\theta^2)} + \frac{2\delta_1(1+\theta)R_3'}{Q^2-(1+\theta)^2} + \frac{2\delta_1(1+\theta)R_3}{Q^2-(1+\theta)^2}\right] \qquad (3.46d)$$

$$R_{13} = \frac{1}{2}\left[-\frac{Q^2 H_1 \tilde{P}}{(Q^2-1)} + \frac{H_2 \alpha_{10} Q^2}{(Q^2-\theta^2)} - \frac{2\delta_1(1-\theta)R_3'}{Q^2-(1-\theta)^2} - \frac{2\delta_1(1-\theta)R_3}{Q^2-(1-\theta)^2}\right] \qquad (3.46e)$$

$$R_{14} = \left[\frac{\alpha\, Q^4 a_{11}\alpha_{10}\tilde{P}}{(Q^2-1)(Q^2-\theta^2)}\right], = \mathrm{R}_{15}\ , \mathrm{R}_{16} = \left[\frac{Q^2 \tilde{P} b_{12}}{2} - \frac{2\delta_1(\theta+Q)R_5'}{\theta(2Q+\theta)} - \frac{2\delta_1(\theta+Q)R_5}{\theta(2Q+\theta)}\right] \qquad (3.46\,f)$$

$$R_{17} = \left[\qquad \frac{Q^2 \tilde{P} b_{12}}{2} - \frac{2\delta_1(\theta-Q)R_5'}{\theta(2Q-\theta)} - \frac{2\delta_1(\theta-Q)R_5}{\theta(2Q-\theta)}\right] \qquad (3.46g)$$

$$R_{18} = \left[\frac{Q^2 \beta_{11} a_{11}}{2} + \frac{Q^4 a_{11}\tilde{P}}{2(Q^2-\theta^2)} + \frac{Q^2 \alpha_{10} b_{12}}{2} - \frac{2\delta_1(1+Q)R_6'}{2Q+1} - \frac{2\delta_1(1+Q)R_6}{2Q+1}\right] \qquad (3.46h)$$

$$R_{19} = \left[\frac{Q^2 \beta_{11} a_{11}}{2} + \frac{Q^{42} \beta_{11}\tilde{P}}{2(Q^2-\theta^2)} - \frac{Q^2 \alpha_{10} b_{12}}{2} + \frac{2\delta_1(1-Q)R_6'}{2Q-1}\frac{2\delta_1(1-Q)R_6}{2Q-1}\right] \qquad (3.46i)$$

$$R_{20} = \frac{\alpha\, Q^4 a_{11}\alpha_{10}}{Q^2-1}, \mathrm{R}_{21} = R_{20}\ , \mathrm{R}_{22} = \frac{\tilde{P} Q^4 a_{11}}{Q^2-\theta^2} \qquad (3.46\,j)$$

We also have

$$R_7(0) = \alpha\left(Q\tilde{P}\right)^2 r_7, r_7 = \frac{1}{2}\left\{Q^4\ r_6 + Q^2 + \frac{1}{Q^2-\theta^2} + Q^2 \delta_0\right\} \qquad (3.47a)$$

$$R_8(0) = \alpha\left(Q\tilde{P}\right)^2 r_8\ , \mathrm{r}_8 = \left[\frac{h_1\, Q^2}{2(Q^2-1)} - \frac{8\delta_0 \delta_1}{Q(Q^2-1)(Q^2-4)} + \frac{4\delta_0 r_1}{Q^2-1}\right] \qquad (3.47b)$$

$$R_9(0) = -\left(Q^2 \tilde{P}\right)^2 r_9\ ,\ \mathrm{r}_9 = \frac{1}{2}\left(\frac{\delta_0}{\tilde{P}} + \alpha\, Q^6 \tilde{P}\right), \mathrm{R}_{10}(0) = \tilde{P}^2 r_{10}\ , \mathrm{r}_{10} = \left[\frac{Q^4 h_2}{Q^2-\theta^2} + \frac{4\delta_1 \theta\, r_2}{Q^2-4\theta^2}\right] \qquad (3.47\mathrm{c})$$

$$R_{11}(0) = \alpha\,(Q^3 \tilde{P})^2 r_{11}\ , \mathrm{r}_{11} = \frac{r_6^2}{2}\left\{\frac{(Q\tilde{P})^2}{(Q^2-\theta^2)}\right\}; \mathrm{R}_{12}(0) = \left(\tilde{P} Q\right)^2 r_{12}, \qquad (3.47d)$$

$$r_{12} = \left[-\frac{Q^2}{2}\left\{\frac{h_1}{(Q^2-1)} + \frac{h_2}{(Q^2-\theta^2)}\right\} + \frac{2\delta_1(1+\theta)\, r_3(\delta_0+1)}{Q^2-(1+\theta)^2}\right] \qquad (3.47e)$$

$$R_{13}(0) = \left(\tilde{P} Q\right)^2 r_{13}, r_{13} = \left[\frac{Q^2}{2}\left\{\frac{h_1}{(Q^2-1)} - \frac{h_2}{(Q^2-\theta^2)}\right\} + \frac{2\delta_1(1-\theta)\, r_3(\delta_0-1)}{Q^2-(1-\theta)^2}\right] \qquad (3.47\,f)$$

$$R_{14}(0) = \left(Q^2 \tilde{P}\right)^3 r_{14}\ , \mathrm{r}_{14} = \frac{\alpha\, r_6}{(Q^2-1)(Q^2+\theta^2)} \qquad , R_{15}(0) = R_{14} \qquad (3.47g)$$

$$R_{16}(0)=(\tilde{P}Q)^2 r_{16}\ ,\ \mathrm{r}_{16}=\left[\frac{h_3}{2}-\frac{2\delta_1(Q+\theta)\ r_6}{\theta(2Q+\theta)}+\frac{2\delta_1(Q+\theta)r_3}{\theta(2Q+\theta)}\right] \qquad (3.47h)$$

$$R_{17}(0)=(\tilde{P}Q)^2 r_{17}\ ,\mathrm{r}_{17}=\left[\frac{h_3}{2}-\frac{2\delta_1(\theta-Q)r_6}{\theta(2Q-\theta)}+\frac{2\delta_1(\theta-Q)r_3}{\theta(2Q-\theta)}\right] \qquad (3.47i)$$

$$R_{18}(0)=(\tilde{P}Q)^2 r_{18}\ ,\ \ \mathrm{r}_{18}=\left[\frac{1}{2}Q^2\delta_0\left\{\ r_6-\frac{1}{Q^2-\theta^2}\right\}-\frac{Q\,h_3}{2}+\frac{2\delta_1(1+Q)r_7}{2Q+1}-\frac{2\delta_1(1+Q)r_6}{2Q+1}\right] \quad (3.47j)$$

$$R_{19}(0)=(\tilde{P}Q)^2 r_{19}\ ,\ \ \mathrm{r}_{19}=\left[\frac{1}{2}Q^2\delta_0\left\{\ r_6-\frac{1}{Q^2-\theta^2}\right\}-\frac{Q\,h_3}{2}+\frac{2\delta_1(1-Q)r_7}{2Q-1}-\frac{2\delta_1(1-Q)r_6}{2Q-1}\right] \qquad (3.47k)$$

$$R_{20}(0)=R_{21}(0)=\alpha(Q^3\tilde{P})^2 r_{20},\ \mathrm{r}_{20}=\frac{r_6}{(Q^2-1)},R_{22}(0)=(Q^3\tilde{P})^2 r_{221}\ ,\mathrm{r}_{221}=-\frac{r_6}{(Q^2-\theta^2)} \quad (3.47l)$$

The solution of (3.45a,b) is

$$\eta^{22}=\ \ \Big[\ \ a_{22}(\tau)\cos Q\,t+b_{22}(\tau)\sin Q\ \mathrm{t}+\frac{\mathrm{R}_7}{\mathrm{Q}^2}+\frac{R_8\sin 2t}{Q^2-4}+\frac{R_9\cos 2t}{Q^2-4}+\frac{R_{10}\sin 2\theta\ \mathrm{t}}{Q^2-4\theta^2}+\frac{R_{11}\cos 2\theta\ \mathrm{t}}{Q^2-4\theta^2}$$

$$+\frac{R_{12}\sin(1+\theta)\mathrm{t}}{\mathrm{Q}^2-(1+\theta)^2}+\frac{R_{13}\sin(\theta-1)\mathrm{t}}{\mathrm{Q}^2-(\theta-1)^2}+\frac{R_{14}\cos(1+\theta)\mathrm{t}}{\mathrm{Q}^2-(1+\theta)^2}+\frac{R_{15}\cos(1-\theta)\mathrm{t}}{\mathrm{Q}^2-(1-\theta)^2}-\frac{R_{16}\sin(Q+\theta)\mathrm{t}}{\theta(2Q+\theta)}$$

$$+\frac{R_{17}\sin(Q-\theta)\mathrm{t}}{\theta(2Q-\theta)}-\frac{R_{18}\sin(Q+1)\mathrm{t}}{2\mathrm{Q}+1}+\frac{R_{18}\sin(1-Q)\mathrm{t}}{2\mathrm{Q}-1}-\frac{\alpha\,R_{20}\cos(Q+1)\mathrm{t}}{2Q+1}+\frac{\alpha\,R_{21}\cos(1-Q)\mathrm{t}}{2Q-1}$$

$$+\frac{R_{22}}{\theta}\left\{\frac{\cos(Q-\theta)\mathrm{t}}{2\mathrm{Q}-\theta}-\frac{\cos(Q+\theta)\mathrm{t}}{2Q+\theta}\right\}\Big] \qquad (3.48a)$$

$$a_{22}(0)=(\tilde{P}Q)^2 r_{22}\ ,\mathrm{r}_{22}=\left[\frac{\alpha\,\mathrm{r}_7}{\mathrm{Q}^2}-\frac{Q^2 r_9}{Q^2-4}+\frac{\alpha\,Q^4 r_6^2 r_{11}}{Q^2-4\theta^2}+\frac{Q^4\tilde{P}\ r_{14}}{\{Q^2-(1+\theta)^2\}}+\frac{Q^4\ \tilde{\mathrm{P}}\mathrm{r}_{14}}{\{Q^2-(1-\theta)^2\}}\right.$$

$$\left.-\frac{\alpha\ \mathrm{Q}^4 r_{20}}{2Q+1}+\frac{\alpha\ \mathrm{Q}^4 r_{20}}{2Q-1}+\frac{Q^4 r_{221}}{\theta}\left\{\frac{1}{2Q-\theta}-\frac{1}{2Q+\theta}\right\}\right] \qquad (3.48b)$$

To evaluate $b_{22}(0)$, we note that

$$\eta^{21}_{,\tau}(0,0)=(\tilde{P}Q)^2 T_{17}\ ,\mathrm{T}_{17}=\left[\frac{r_{221}}{Q^2}-\frac{\delta_0}{Q^2-1}-\frac{2\delta_0}{Q(Q^2-4)(Q^2-1)}+\frac{\mathrm{r}_6\delta_1}{\theta}\left\{\frac{1}{2Q-\theta}-\frac{1}{2Q+\theta}\right\}\right.$$

$$\left.+r_7\left\{\frac{1}{2Q+1}-\frac{1}{2Q-1}\right\}\right] \qquad (3.48c)$$

Therefore, the evaluation of $b_{22}(0)$ follows from the second of (3.45b) to yield

$$b_{22}(0)=\tilde{P}^2 Q T_{18}\ ,\mathrm{T}_{18}=-\left[\frac{2\ r_8}{Q^2-4}+\frac{2\ \theta\ \mathrm{r}_{10}}{\mathrm{Q}^2(Q^2-4\theta^2)}+\frac{(1+\theta)r_{12}}{Q^2-(1+\theta)^2}+\frac{(\theta-1)r_{13}}{Q^2-(\theta-1)^2}\right.$$

$$\left.-\frac{(Q+\theta)r_{16}}{\theta(2Q+\theta)}+\frac{(Q-\theta)\mathrm{r}_{17}}{\theta(2\mathrm{Q}-\theta)}-\frac{(Q+1)\mathrm{r}_{18}}{2\mathrm{Q}+1}+\frac{(1-Q)\mathrm{r}_{19}}{2\mathrm{Q}-1}+\frac{\mathrm{T}_{17}}{\tilde{\mathrm{P}}}\right] \qquad (3.49)$$

Thus we eventually have the following:

$$\xi_0=\in\left\{\zeta^{10}+\zeta^{11}\overline{\xi}+\zeta^{12}\overline{\xi}^2+\cdots\right\}+\in^2\left\{\zeta^{20}+\zeta^{21}\overline{\xi}+\zeta^{22}\overline{\xi}^2+\cdots\right\}+\cdots \qquad (3.50a)$$

$$\xi_1=\in\left\{\eta^{11}\overline{\xi}+\eta^{12}\overline{\xi}^2+\cdots\right\}+\in^2\left\{\eta^{21}\overline{\xi}+\eta^{22}\overline{\xi}^2+\cdots\right\}+\cdots \qquad (3.50b)$$

3.1 Maximum displacement

We shall now determine the maximum displacements $\xi_{0\max}$ and $\xi_{1\max}$ of $\xi_0(t)$ and $\xi_1(t)$ respectively. For $\xi_{0\max}$, the condition is

$$\xi_{0,t}(t_a,\tau_a)+\bar{\xi}\xi_{0,\tau}(t_a,\tau_a)=0 \qquad (3.51)$$

where t_a and τ_a are the critical values of t and τ. We let

$$t_a=t_0+\bar{\xi}\,t_{01}+\in\left(t_{10}+\bar{\xi}\,t_{11}\right)+\ldots\,;\ \tau_a=\bar{\xi}\,t_a=\bar{\xi}\left\{t_0+\bar{\xi}\,t_{01}+\in\left(t_{10}+\bar{\xi}\,t_{11}\right)+\ldots\right\} \qquad (3.52)$$

By substituting (3.52) into (3.51) and equating the coefficients of $\in$, $\in\bar{\xi}$ and $\in^2$, we get the following respective equations

$$\zeta^{10}_{,t}(t_0.0)=0,\quad t_{01}\zeta^{10}_{,tt}+\zeta^{11}_{,t}+\zeta^{10}_{,\tau}=0\ ,\ t_{10}\zeta^{10}_{,tt}+\zeta^{10}_{,t}=0 \qquad (3.53)$$

where (3.53) is evaluated at $(t_0,0)$. The first of (3.53) is evaluated to give

$$\sin t_0-\theta\sin\theta\,t_0=0 \qquad (3.54a)$$

By maintaining just the first two terms in the expansions in (3.54a), we have

$$t_0\cong\sqrt{\frac{6}{1+\theta^4}} \qquad (3.54b)$$

From the second equation in (3.53), we have

$$t_{01}=-\left.\frac{\left(\zeta^{11}_{,tt}+t_0\zeta^{10}_{,\tau}\right)}{\zeta^{10}_{,tt}}\right|_{(t_0,0)}=\frac{(\delta_0+t_0)\cos t_0}{\cos t_0-\theta^2\cos\theta\,t_0} \qquad (3.54c)$$

From the third equation in (3.53), we get

$$t_{10}=-\left.\frac{\zeta^{21}_{,t}}{\zeta^{10}_{,tt}}\right|_{(t_0,0)}=0 \qquad (3.54d)$$

The maximum, displacement $\xi_{0\max}$ is obtained by evaluating (3.50a) at (t_a,τ_a) using (3.54a-d) to get

$$\xi_{0\max}=\in\left[\ \zeta^{10}+\bar{\xi}\left(t_0\zeta^{10}_{,\tau}+\zeta^{11}\right)+\bar{\xi}^2\left(t_{01}\zeta^{10}_{,\tau}+t_{01}\zeta^{11}_{,t}+t_0\zeta^{11}_{,\tau}+\zeta^{12}\right)\right]+\in^2\bar{\xi}^2\zeta^{22}$$
$$+O\left(\in\bar{\xi}^3\right)+O\left(\in^2\bar{\xi}^3\right) \qquad (3.55)$$

where (3.55) is evaluated at $(t_0,0)$. On substituting for terms in (3.55) and simplifying, we get

$$\xi_{0\max}=\tilde{P}\in\left[T_7+\bar{\xi}(\sin t_0-t_0\cos t_0)+2\bar{\xi}^2t_0\delta_0^2\sin t_0\right]+\frac{K_0}{\lambda_C}\left(Q^2\tilde{P}\right)^2T_8\bar{\xi}^2\in^2$$
$$+O\left(\in\bar{\xi}^3\right)+O\left(\in^2\bar{\xi}^3\right) \qquad (3.56a)$$

where

$$T_7=(\cos\theta t_0-\cos t_0),\ T_8=\left[-r_{30}\cos t+r_{23}+\frac{r_{24}\cos 2Q\,t}{1-4Q^2}+\frac{r_6}{Q}\left\{\frac{\cos(1+Q)\,t}{2+Q}-\frac{\cos(1-Q)\,t}{2-Q}\right\}\right.$$
$$+r_{26}\left\{\frac{\cos(Q-\theta)\,t}{1-(Q-\theta)^2}+\frac{\cos(Q+\theta)\,t}{1-(Q+\theta)^2}\right\}-\frac{r_{27}\cos 2t}{3}+\frac{r_{28}\cos 2\theta\,t}{Q^4(1-4\theta^2)}+\frac{r_{29}}{\theta Q^3\tilde{P}}\left\{\frac{\cos(1-\theta)\,t}{2-\theta}-\frac{\cos(1+\theta)\,t}{2+\theta}\right\}$$
$$\left.\left.-\frac{2\cos Qt}{Q^2\tilde{P}(1-Q^2)}+\frac{2\cos\theta\,t}{Q^2\tilde{P}(Q^2-\theta^2)}\right]\right|_{(t_0,0)} \qquad (3.56b)$$

The condition for determining $\xi_{1\max}$ is

$$\xi_{1,t}(t_c,\tau_c)+\bar{\xi}_{1,\tau}(t_c,\tau_c)=0 \quad (3.57a)$$

where t_c and τ_c are the values of t and τ respectively at $\xi_{1\max}$. We assume the following series

$$t_c=\tilde{t}_0+\bar{\xi}\,\tilde{t}_{01}+\in\left(\tilde{t}_{10}+\bar{\xi}\,\tilde{t}_{11}\right)+\ldots\,;\ \tau_c=\bar{\xi}\,t_c=\bar{\xi}\left\{\tilde{t}_0+\bar{\xi}\,\tilde{t}_{01}+\in\left(\tilde{t}_{10}+\bar{\bar{\xi}}\,t_{11}\right)+\ldots\right\} \quad (3.57b)$$

The introduction of (3.51) into (3.57a), using (3.57b) and equating of the coefficients of $\in$,$\in\bar{\xi}$ and $\in^2$, yield the following respective equations

$$\eta^{11}_{,t}=0\ ;\ \tilde{t}_{01}\eta^{11}_{,tt}+\tilde{t}_0\,\eta^{11}_{,t\tau}+\eta^{11}_{,\tau}=0\ ;\ \tilde{t}_{10}\eta^{11}_{,tt}+\eta^{21}_{,t}=0 \quad (3.57c)$$

From the first of (3.57c), we have

$$a_{11}(0)\sin Q\,\tilde{t}_0+Q\left\{\frac{\alpha_{10}(0)}{Q^2-1}+\frac{\tilde{P}\,\theta\sin\theta\,t}{(Q^2-\theta^2)}\right\}=0 \quad (3.58a)$$

An approximate value of $\tilde{t}_0$ from (3.58a) is

$$\tilde{t}_0\cong\sqrt{\frac{6\left\{Q^2r_6+\left(\frac{1}{Q^2-1}+\frac{\theta^2}{Q^2-\theta^2}\right)\right\}}{Q^4r_6-\left(\frac{\theta^4}{Q^2-\theta^2}+\frac{1}{Q^2-1}\right)}} \quad (3.58b)$$

From the second equation in (3.57c), we have

$$\tilde{t}_{01}=-\left(\frac{\tilde{t}_0\eta^{11}_{,t\tau}+\eta^{12}_{,t}+\eta^{11}_{,\tau}}{\eta^{11}_{,tt}}\right)\Bigg|_{(t_0,0)}=\left(\frac{T_2\tilde{t}_0-T_4-T_5}{T_1}\right) \quad (3.59a)$$

where

$$\eta^{11}_{,tt}(\tilde{t}_0,0)=\tilde{P}Q^2T_1,\ \mathrm{T}_1=\left[\ Q^2r_6\cos Q\,\tilde{\mathrm{t}}_0+\frac{\cos\tilde{t}_0}{Q^2-1}-\frac{\theta^2\cos\theta\,\tilde{\mathrm{t}}_0}{\mathrm{Q}^2-\theta^2}\right] \quad (3.59b)$$

$$\eta^{11}_{,t\tau}(\tilde{t}_0,0)=-\tilde{P}Q^2T_2\ ,\ \mathrm{T}_2=\left(\ \delta_1r_6\sin Q\,\tilde{\mathrm{t}}_0+\frac{\delta_0\sin\tilde{t}_0}{Q^2-1}\right) \quad (3.59c)$$

$$b_{12}(0)=\tilde{P}QT_3,\ T_3=\left[\frac{h_1}{Q^2-1}-\frac{h_2}{Q^2-\theta^2}-\mathrm{r}_6-\frac{\delta_0}{Q^2-1}\right] \quad (3.59d)$$

From the

$$\eta^{12}_{,t}(\tilde{t}_0,0)=\tilde{P}Q^2T_{4,}\,T_4=\left[T_3\cos Q\,\tilde{t}_0-\frac{h_1}{Q^2-1}+\frac{h_2\theta}{Q^2-\theta^2}\right], \quad (3.59e)$$

$$\eta^{11}_{,\tau}(\tilde{t}_0,0)=\tilde{P}Q^2T_5,\quad T_5=\left(\ \delta_1r_6\cos Q\,\tilde{\mathrm{t}}_0+\frac{\delta_0\cos\tilde{t}_0}{Q^2-1}\right) \quad (3.59f)$$

third equation in (3,57c), we have

$$\tilde{t}_{10}=-\frac{\eta^{21}_{,t}}{\eta^{11}_{,tt}}\Bigg|_{(\tilde{t}_0,0)}=-\frac{\tilde{\mathrm{P}}\mathrm{T}_6}{\mathrm{T}_1} \quad (3.59g)$$

$$_{,t}^{21}(\tilde{t}_0,0)=\left(\tilde{P}Q^2\right)^2 T_6\ ;\ \mathrm{T}_6=\left[\frac{r_{77}\sin Q\,\mathrm{t}}{\mathrm{Q}}-\frac{2r_1\sin 2t}{Q^2-4}-\frac{2\theta\, r_2\sin 2\theta\,\mathrm{t}}{\mathrm{Q}^2\left(Q^2-4\theta^2\right)}-\frac{(1+\theta)\,r_3\sin(1+\theta)\,\mathrm{t}}{\mathrm{Q}^2-(1+\theta)^2}\right.$$

$$+\frac{(1-\theta)\,r_3\sin(1-\theta)\,\mathrm{t}}{\mathrm{Q}^2-(1-\theta)^2}-\frac{\mathrm{r}_5}{\theta}\left\{\frac{(\theta+Q)\sin(\theta+Q)\,\mathrm{t}}{2\mathrm{Q}+\theta}-\frac{(\theta-Q)\sin(\theta-Q)\,\mathrm{t}}{2\mathrm{Q}-\theta}\right\}\ \mathrm{On}$$

$$\left.\left.+\frac{\mathrm{r}_5}{\theta}\left\{\frac{(1+Q)\sin(\theta+Q)\,\mathrm{t}}{2\mathrm{Q}+1}-\frac{(1-Q)\sin(\theta-Q)\,\mathrm{t}}{2\mathrm{Q}-1}\right\}\right]\right|_{(\tilde{\mathrm{t}}_0,0)} \qquad (3.59h)$$

evaluating (3.50b) at (t_c,τ_c), using(3.58b) and (3.59a-h), we have

$$\xi_{1\max}=\in\left[\bar{\xi}\eta^{11}+\bar{\xi}^2\left(\tilde{t}_0\eta^{11}_{,\tau}+\eta^{12}\right)\right]\Big|_{(\tilde{t}_0,o)}+\in^2\left[\bar{\xi}\eta^{21}+\bar{\xi}^2\left(\tilde{t}_{10}\eta^{11}_{,\tau}+\tilde{t}_{01}\tilde{t}_{10}\eta^{11}_{,tt}+\tilde{t}_{10}\tilde{t}_0\eta^{11}_{,t\tau}\right.\right.$$

$$\left.\left.+\tilde{t}_{10}\eta^{12}_{,t}+\tilde{t}_{01}\eta^{21}_{,t}+\tilde{t}_0\eta^{21}_{,\tau}+\eta^{22}\right)\right]\Big|_{(\tilde{t}_0,o)}+O\left(\in\bar{\xi}^3\right)+O\left(\in^2\bar{\xi}^3\right) \qquad (3.60)$$

where terms not included in (3.60) will automatically vanish on evaluation. Some of the major terms in (3.60) are evaluated as follows:

$$\eta^{11}(t_0,0)=\tilde{P}Q^2T_9\,,\ \mathrm{T}_9=\left[\frac{\cos\theta\,\mathrm{t}_0}{Q^2-\theta^2}-\frac{\cos\mathrm{t}_0}{Q^2-1}-r_6\cos Q\,\mathrm{t}_0\right] \qquad (3.61a)$$

$$\eta^{11}_{,\mathrm{t}}(t_0,0)=\tilde{P}Q^2T_{10}\,,\ \mathrm{T}_{10}=\left[r_6\delta_1\cos Q\,\mathrm{t}_0+\frac{\cos\mathrm{t}_0}{Q^2-1}\right] \qquad (3.61b)$$

$$\eta^{12}_{,\mathrm{t}}(t_0,0)=\tilde{P}Q^2T_{11}\,,\ \mathrm{T}_{11}=\left[\frac{h_3\sin Q\,\mathrm{t}_0}{Q^2}-\frac{h_1\sin\mathrm{t}_0}{Q^2-1}+\frac{h_2\sin\theta\,\mathrm{t}_0}{Q^2-\theta^2}\right] \qquad (3.61c)$$

$$\eta^{21}(t_0,0)=(\tilde{P}Q^2)^2T_{12}\,,\ \mathrm{T}_{12}=\left[\frac{r_0}{Q^2}-r_{77}\cos Q\,\mathrm{t}+\frac{\mathrm{r}_1\cos 2t}{Q^2-4}+\frac{r_2\cos 2\theta\,\mathrm{t}}{\left(Q^2-4\theta^2\right)}+\frac{r_3\cos(1+\theta)\,\mathrm{t}}{Q^2-(1+\theta)^2}\right.$$

$$\left.\left.+\frac{r_4\cos(1-\theta)\,\mathrm{t}}{Q^2-(1-\theta)^2}-\frac{r_5}{\theta}\left\{\frac{\cos(Q+\theta)\,\mathrm{t}}{2\mathrm{Q}+\theta}-\frac{\cos(\theta-Q)\,\mathrm{t}}{2\mathrm{Q}\theta}\right\}-\mathrm{r}_6\left\{\frac{\cos(1-Q)}{2Q-1}-\frac{\cos(1+Q)}{2Q+1}\right\}\right]\right|_{(\mathrm{t}_0,0)} \qquad (3.61d)$$

$$\eta^{21}_{,\tau}(\tilde{t}_0,0)=\left(\tilde{P}Q\right)^2T_{16}\,,\ \mathrm{T}_{16}=\left[Q^2\delta_1 r_{77}\cos Q\,\mathrm{t}-\frac{\delta_0}{\mathrm{Q}^2-1}-\frac{2\delta_0\cos 2t}{Q\left(Q^2-4\right)\left(Q^2-1\right)}+\frac{\delta_0 r_3\cos(1+\theta)\,\mathrm{t}}{\mathrm{Q}^2-(1+\theta)^2}\right.$$

$$\left.\left.+\frac{\mathrm{r}_6\delta_1}{\tilde{P}\theta}\left\{\frac{\cos(\theta-Q)\,\mathrm{t}}{2Q-\theta}-\frac{\cos(\theta+Q)\,\mathrm{t}}{2Q+\theta}\right\}\right]\right|_{(\tilde{t}_0 0)} \qquad (3.61e)$$

$$\eta^{22}(\tilde{t}_0,0)=\left(\tilde{P}Q\right)^2T_{19}\,,\ \mathrm{T}_{19}=\left[r_{22}\cos Q\,\mathrm{t}+\frac{\mathrm{T}_{18}\sin Q\,\mathrm{t}}{Q}+\frac{\alpha\,\mathrm{r}_7}{\mathrm{Q}^2}+\frac{r_8\sin 2\,\mathrm{t}}{\mathrm{Q}^2 4}-\frac{Q^2 r_9\cos 2t}{Q^2-4}+\frac{r_{10}\sin 2\theta\,\mathrm{t}}{Q^2-4\theta^2}\right.\quad \mathrm{On}$$

$$+\frac{\mathrm{r}_6^2Q^2\alpha\, r_{11}\cos 2\theta\,\mathrm{t}}{Q^2-4\theta^2}+\frac{r_{12}\sin(1+\theta)\,\mathrm{t}}{\mathrm{Q}^2-(1+\theta)^2}+\frac{r_{13}\sin(\theta-1)\,\mathrm{t}}{\mathrm{Q}^2-(\theta-1)^2}+\frac{\tilde{P}Q^4r_{14}\cos(\theta+1)\,\mathrm{t}}{\left\{\mathrm{Q}^2-(\theta+1)^2\right\}}+\frac{\tilde{P}Q^4r_{14}\cos(1-\theta)\,\mathrm{t}}{\left\{\mathrm{Q}^2-(1-\theta)^2\right\}}$$

$$-\frac{r_{16}\sin(Q+\theta)\,\mathrm{t}}{\theta\,(2Q+\theta)}+\frac{r_{17}\sin(Q-\theta)\,\mathrm{t}}{\theta\,(2Q-\theta)}-\frac{r_{18}\sin(Q+1)\,\mathrm{t}}{(2Q+1)}+\frac{r_{19}\sin(1-Q)\,\mathrm{t}}{\theta(2Q-1)}-\frac{\alpha\ \mathrm{r}_{20}\cos(Q+1)\,\mathrm{t}}{\tilde{\mathrm{P}}(2Q+1)}$$

$$\left.\left.+\frac{\alpha\ \mathrm{r}_{20}\cos(1-Q)\,\mathrm{t}}{\tilde{\mathrm{P}}(2Q-1)}+\frac{r_{221}}{\theta}\left\{\frac{\cos(Q-\theta)\,\mathrm{t}}{2\mathrm{Q}-\theta}-\frac{\cos(Q+\theta)\,\mathrm{t}}{2\mathrm{Q}+\theta}\right\}\right]\right|_{(\tilde{t}_0,0)} \qquad (3.61f)$$

substituting all the relevant terms into (3.60) and simplifying, we have

$$\xi_{1\max} = \in \left[\tilde{P}Q^2\bar{\xi}T_9 + \bar{\xi}^2\left(\tilde{t}_0\tilde{P}Q^2T_{10} + \tilde{P}Q^2T_{11}\right)\right] + \in^2 \Big[\left(\tilde{P}Q^2\right)^2 T_{12}\bar{\xi} + \bar{\xi}^2\Big\{\tilde{t}_{10}\tilde{P}Q^2T_{10} + \tilde{t}_{10}\tilde{t}_{01}\tilde{P}Q^2T_1 - \tilde{t}_{10}\tilde{t}_0\tilde{P}Q^2T_2 + \tilde{t}_{10}\tilde{P}Q^2T_4 + \tilde{t}_{01}\tilde{P}Q^2T_6 + \tilde{t}_0\left(\tilde{P}Q\right)^2T_{16} + \left(\tilde{P}Q\right)^2T_{19}\Big\}\Big] + O\left(\in\bar{\xi}^3\right) + O\left(\in^2\bar{\xi}^3\right) \quad (3.62)$$

From (3.4), the net maximum displacement ξ_m comes from (3.56a) and (3.62) as

$$\xi_m = C_1 \in + C_2 \in^2 + \cdots, \quad C_1 = \tilde{P}\left[T_7 + \bar{\xi}\left\{(\sin t_0 - t_0\cos t_0) + Q^2T_9\right\} + \bar{\xi}^2\left\{2t_0\delta_0\sin t_0 + \tilde{t}_0Q^2T_{10} + Q^2T_{11}\right\}\right] \quad (3.63a)$$

$$C_2 = \left[\left(\tilde{P}Q^2\right)^2\bar{\xi}T_{12} + \bar{\xi}^2\left\{\frac{K_0\left(\tilde{P}Q^2\right)^2T_8}{\lambda_C} + \tilde{t}_{10}\tilde{P}Q^2T_{10} + \tilde{t}_{10}\tilde{t}_{01}\tilde{P}Q^2T_1 - \tilde{t}_{10}\tilde{t}_0\tilde{P}Q^2T_2 + \tilde{t}_{10}\tilde{P}Q^2T_4 + \tilde{t}_{01}\tilde{P}Q^2T_6 + \tilde{t}_0(\tilde{P}Q)^2T_{16} + (\tilde{P}Q)^2T_{19}\right\}\right] \quad (3.63b)$$

3.2 Dynamic buckling load

As in [1-5], the dynamic buckling load λ_D is obtained from the maximization in (3.4) .The usual procedure [5] is to, first of all , reverse the series(3.63a) so that we have

$$\in = d_1\xi_m + d_2\xi_m^2 + \cdots \quad (3.64a)$$

By substituting into (3.64a) for ξ_m from (3.63a) and equating the coefficients of $\in$ and $\in^2$, we have the following respective values:

$$d_1 = \frac{1}{C_1}, \ d_2 = -\frac{C_2}{C_1^3} \quad (3.64b)$$

The maximization (3.4) easily follows, through (3.64a) to give

$$\xi_m(\lambda_D) \equiv \xi_{mD} = -\frac{d_1}{2d_2} = \frac{C_1^2}{2C_2} \quad (3.64c)$$

where ξ_{mD} is the value of ξ_m at buckling where $\lambda = \lambda_D$. On evaluating (3.64a) at $\lambda = \lambda_D$, we have

$$\in_D = \frac{C_1}{4C_2} \quad (3.65)$$

where $\in_D$ is the value of $\in$ at $\lambda = \lambda_D$. On substituting for $\in_D$ into (3.65), we get the dynamic buckling load λ_D as

$$\lambda_D\left(\frac{\omega_1}{\omega_0}\right)^2 = \frac{C_1}{4C_2} \quad (3.66)$$

4.0 Special result; the case $\theta = 1$

The value $\theta = 1$ is one of the cases where the above analysis fails because there is a high level of parametric resonance in the equations of the least orders of the perturbation parameters leading to unbounded solution. In this case, the frequency of the forcing function is equal to that of the natural vibration of the unloaded structure. An approximate but fairly straightforward and accurate solution can be obtained by disregarding the pre-buckling inertia as well as the viscous damping on the pre-buckling mode so that, from (3.1) and (3.2), we have

$$\xi_0(t) \cong \frac{K_0}{\lambda_C}\xi_1\left(\xi_1 + 2 + \bar{\xi}\right) + \in P\cos t \quad (4.1)$$

$$\frac{d^2\xi_1}{dt^2} + 2\bar{\xi}\delta_1\frac{d\xi_1}{dt} + Q^2\xi_1(1 - \in P\cos t) - Q^2\xi_1^2\left(\alpha + \frac{K_0\bar{\xi}}{\lambda_C}\right) = Q^2 \in \bar{\xi}P\cos t \quad (4.2)$$

$$\xi_1(0) = \frac{d\xi_1(0)}{dt} = 0 \quad (4.3)$$

where (4.2) is obtained by substituting for $\xi_0(t)$ from (4.1) into (3.2).Using the fact that $\frac{d\xi_1}{dt} = \xi_{1,t} + \bar{\xi}\xi_{1,\tau}$, we now solve (4.1) and (4.3) by letting

$$\xi_1(t) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} A^{ij}(t,\tau)\in^i \bar{\xi}^j \qquad (4.4)$$

where the ij, as in A^{ij}, are superscripts and not powers. On substituting into (4.2a), we get

$$SA^{11} \equiv A^{11}_{,tt} + Q^2 A^{11} = Q^2 P\cos t \qquad (4.5)$$

$$SA^{12} = -2A^{11}_{,t\tau} - 2\delta_1 A^{11}_{,t} \qquad (4.6)$$

$$SA^{21} = -A^{11}\left(Q^2 P\cos t\right) \qquad (4.7)$$

$$SA^{22} = -A^{12}\left(Q^2 P\cos t\right) + \alpha Q^2\left(A^{11}\right)^2 - 2A^{21}_{,t\tau} - 2\delta_1 A^{21}_{,t} \qquad (4.8)$$

The initial conditions are evaluated at $(t,\tau) = (0,0)$ as

$$A^{ij} = 0, \text{for } i = 1,2,3,\cdots;\ j = 1,2,3,\cdots; A^{11}_{,t} = 0, A^{1k}_{,t} + A^{1p}_{,\tau} = 0; A^{21}_{,t} = 0$$
$$A^{2k}_{,t} + A^{2p}_{,\tau} = 0; p = k-1, k = 2,3,4,\cdots \qquad (4.9)$$

On solving (4.5),using appropriate initial conditions, we have

$$A^{11}(t,\tau) = l_{11}(\tau)\cos Q t + l_{12}(\tau)\sin Q t + \frac{Q^2 P\cos t}{Q^2-1};\ l_{11}(0) = -\frac{Q^2 P}{Q^2-1},\quad l_{12}(0) = 0 \qquad (4.10)$$

We now substitute into (4.6), and to ensure a uniformly valid solution in t, we have

$$l'_{11} + \delta_1 l_{11} = 0\ ,\quad \text{and}\quad l'_{12} + \delta_1 l_{12} = 0 \qquad (4.11a)$$

The solutions of (4.11a) are

$$l_{11} = l_{11}(0)e^{-\delta_1\tau}\ , l_{12} = 0 \qquad (4.11b)$$

Therefore we have

$$A^{11}(t,\tau) = l_{11}\cos Q t + \frac{Q^2 P\cos t}{Q^2-1} \qquad (4.11c)$$

The remaining equation in (4.6) is solved to get

$$A^{12}(t,\tau) = l_{21}(\tau)\cos Q t + l_{22}(\tau)\sin Q t + \frac{2\delta_1 Q^2 P\sin t}{Q^2-1}, l_{21}(0) = 0,\ l_{22}(0) = -\frac{QP\delta_1(1+Q)}{\left(Q^2-1\right)} \qquad (4.12)$$

On substituting into (4.7) and solving the equation, we have

$$A^{21}(t,\tau) = l_{24}(\tau)\cos Q t + l_{25}(\tau)\sin Q t + \frac{Q^2 P}{2}\left[l_{11}\left\{\frac{\cos(Q-1)t}{2Q-1} - \frac{\cos(Q+1)t}{2Q+1}\right\}\right.$$
$$\left. + \frac{PQ^2}{2}\left\{\frac{1}{Q^2} + \frac{\cos 2t}{Q^2-4}\right\}\right] \qquad (4.13a)$$

If we

$$l_{24}(0) = -\left(PQ^2\right)^2 q_0\ , q_0 = \left[\frac{Q^2-2}{Q^2\left(Q^2-4\right)} - \frac{2}{\left(Q^2-1\right)\left(4Q^2-1\right)}\right]\ , l_{25}(0) = 0 \qquad (4.13b)$$

substitute into (4.8) and ensure a uniformly valid solution in t by equating to zero the coefficients of *cosQt* and *sinQt* , we get the following respective equations

$$l'_{25} + \delta_1 l_{25} = 0 \text{ and } l'_{24} + \delta_1 l_{24} = 0 \qquad (4.14a)$$

On solving (4.14a) we have

$$l_{25}(\tau)=l_{25}(0)e^{-\delta_1\tau}=0\ , l_{24}(0)e^{-\delta_1\tau} \qquad (4.14b)$$

If we re-arrange the remaining terms in the substitution into (4.8),we have

$$SA^{22}=p_1+p_2\cos 2t+p_3\{\cos(Q+1)\,t-\cos(Q-1)\,t\}+p_4\sin(Q+1)\,t+p_5\sin(Q-1)\,t$$
$$+P_6\sin 2t \qquad (4.15a)$$

$$A^{22}(0,0)=0\,, A^{22}_{,t}(0,0)+A^{21}_{,\tau}(0,0)=0 \qquad (4.15b)$$

$$p_1=\frac{\alpha Q^2}{2}\left\{l_{11}^2+\left(\frac{PQ^2}{Q^2-1}\right)^2\right\}, p_2=\frac{\alpha}{2}\left(\frac{PQ^2}{Q^2-1}\right)^2, p_3=\alpha PQ^2 l_{11} \qquad (4.16a)$$

$$p_4=\left[\frac{PQ^2 l_{22}}{2}-\frac{PQ^2 l'_{11}(!+Q)}{2Q+1}-\frac{PQ^2 l_{11}(!+Q)}{2Q+1}\right], p_5=\left[\frac{PQ^2 l_{22}}{2}-\frac{PQ^2 l'_{11}(Q-1)}{2Q-1}-\frac{PQ^2 l_{11}(!Q-1)}{2Q+1}\right] \quad (4.16b)$$

$$p_1(0)=\alpha\left(PQ^2\right)q_1\,, q_1=\left(\frac{Q}{Q^2-1}\right)^2, p_2(0)=\alpha\left(PQ^2\right)q_2\ \ , q_2=\frac{1}{2}\left(\frac{Q}{Q^2-1}\right)^2 \qquad (4.16c)$$

$$p_3(0)=\alpha\left(PQ^2\right)q_3\,, q_3=q_1\,, p_4(0)=\alpha\left(PQ^2\right)q_4\,, q_4=-\left[\frac{\delta_1(1+Q^2)}{2Q(Q^2-1)^2}+\frac{(Q+1)(1-\delta_1)}{(2Q+1)(Q^2-1)}\right] \quad (4.16d)$$

$$p_5(0)=\alpha\left(PQ^2\right)q_5\,, q_5=-\left[\frac{\delta_1(1+Q^2)}{2Q(Q^2-1)^2}+\frac{(Q-1)(1-\delta_1)}{(2Q-1)(Q^2-1)}\right] \qquad (4.16e)$$

On solving (4.15a,b), we have

$$A^{22}(t,\tau)=[\,l_{26}\cos Q\,t+l_{27}\sin Q\,t+\frac{p_1}{Q^2}+\frac{p_2\cos 2t}{Q^2-4}+\frac{p_2\sin 2t}{Q^2-4}+p_3\left\{\frac{\cos(Q-1)\,t}{2Q-1}-\frac{\cos(Q+1)\,t}{2Q+1}\right\}$$
$$-\frac{p_4\sin(Q+1)\,t}{2Q+1}+\frac{p_5\sin(Q-1)\,t}{2Q-1}+\frac{p_6\sin 2t}{Q^2-1}\Bigg] \qquad (4.17)$$

$$l_{26}(0)=-\alpha\left(PQ^2\right)^2 q_7\,, q_7=\left[\frac{q_1}{Q^2}+\frac{q_2}{Q^2-4}+\frac{2q_3}{4Q^2-1}\right] \qquad (4.18a)$$

$$l_{27}(0)=-\frac{1}{Q}\left[\frac{2p_6}{Q^2-4}-\frac{p_4(1+Q)}{2Q+1}+\frac{(Q-1)p_5}{2Q-1}+l'_{24}+\frac{pQ^2}{2}l'_{11}\left\{\frac{1}{2Q-1}-\frac{1}{2Q+1}\right\}\right]\Bigg|_{\tau=0} \qquad (4.18b)$$

On further simplifying (4.18b) we have

$$l_{27}(0)=-\left(PQ^2\right)q_8\,, q_8=\frac{1}{Q}\left[\frac{2\,q_6}{Q^2-4}-\frac{(Q+1)\,q_4}{2Q+1}+\frac{(Q-1)\,q_4}{2Q-1}+\delta_1 q_0+\frac{\delta_1}{(Q^2-1)(4Q^2-1)}\right] \qquad (4.19)$$

Thus we have

$$\xi_1(t)=\in\left(A^{11}\overline{\xi}+A^{12}\overline{\xi}^2+\cdots\right)+\in^2\left(A^{21}\overline{\xi}+A^{22}\overline{\xi}^2+\cdots\right)+\cdots \qquad (4.20)$$

We let $\hat{t}$ and $\hat{\tau}$ be the values of t and τ respectively at the maximum displacement ξ_{1M} of equation (4.20).The condition for ξ_{1M} is

$$\xi_{1,t}(\hat{t},\hat{\tau})+\overline{\xi}\xi_{1,\tau}(\hat{t},\hat{\tau})=0 \qquad (4.21)$$

We let

$$\hat{t}=\hat{t}_0+\overline{\xi}\,\hat{t}_{01}+\in\left(\hat{t}_{10}+\overline{\xi}\,\hat{t}_{11}\right)+\cdots,\ \hat{\tau}=\overline{\xi}\,\hat{t}=\overline{\xi}\left\{\hat{t}_0+\overline{\xi}\,\hat{t}_{01}+\in\left(\hat{t}_{10}+\overline{\xi}\,\hat{t}_{11}\right)+\cdots\right\} \qquad (4.22)$$

We substitute (4.20) into (4.21), using (4.22), and equate the coefficients of $\in\bar{\xi}$, $\in\bar{\xi}^2$ and $\in^2\bar{\xi}$ to get the following respective equations evaluated at $(\hat{t}_0,0)$:

$$A^{11}_{,\mathrm{t}}=0\ ,\quad \hat{\mathrm{t}}_{01}A^{11}_{,\mathrm{t\,t}}+A^{12}_{,\mathrm{t}}+A^{11}_{,\tau}=0,\quad \hat{\mathrm{t}}_{10}A^{11}_{,\mathrm{t\,t}}+A^{21}_{,\mathrm{t}}=0 \qquad (4.23)$$

From the first of (4.23) we get

$$Q\sin Q\hat{t}_0-\sin\hat{t}_0=0 \qquad (4.24a)$$

An approximate value of $\hat{t}_0$ from (4.24a) is

$$\hat{t}_0\cong\sqrt{\frac{6}{1+Q^2}} \qquad (4.24b)$$

We note the following simplifications

$$A^{11}_{,\mathrm{t}}(\hat{t}_0,0)=PQ^2T_{20}\ ,\ \mathrm{T}_{20}=\frac{1}{Q^2-1}\left(Q^2\cos Q\,\hat{\mathrm{t}}_0-\cos\hat{t}_0\right);\ A^{11}_{,\tau}(\hat{t}_0,0)=\frac{\delta_1PQ^2\cos Q\,\hat{\mathrm{t}}_0}{Q^2-1} \qquad (4.25a)$$

$$A^{12}_{,\mathrm{t}}(\hat{t}_0,0)=PQ^2T_{21}\ ,\ \mathrm{T}_{21}=\frac{\delta_1}{Q^2-1}\left[2\cos\hat{t}_0-\left(1+Q^2\right)\cos Q\,\hat{\mathrm{t}}_0\right] \qquad (4.25b)$$

$$A^{21}_{,\mathrm{t}}(\hat{t}_0,0)=PQ^2T_{22}\ ,\ \mathrm{T}_{22}=\left[Qq_0\sin Q\,\mathrm{t}+\frac{1}{2}\left\{\left\{\frac{1}{\mathrm{Q}^2-1}\left\{\begin{matrix}\dfrac{(Q-1)\sin(Q-1)\,\mathrm{t}}{2Q-1}\\ -\dfrac{(Q+1)\sin(Q+1)\,\mathrm{t}}{2Q+1}\end{matrix}\right\}-\frac{\sin 2t}{Q^2-4}\right\}\right\}\right]_{(\hat{\mathrm{t}}_0,0)} \qquad (4.25c)$$

$$A^{21}_{,\tau}(\hat{t}_0,0)=\left(PQ^2\right)^2T_{23}\,,\ \mathrm{T}_{23}=\left[q_0\delta_1\cos Q\hat{t}_0+\frac{1}{2\left(Q^2-1\right)}\left\{\frac{\cos(Q-1)\hat{\mathrm{t}}_0}{2\mathrm{Q}-1}-\frac{\cos(Q+1)\hat{\mathrm{t}}_0}{2\mathrm{Q}+1}\right\}\right] \qquad (4.25d)$$

$$A^{22}(\hat{t}_0,0)=\left(PQ^2\right)^2T_{24}\,,\ \mathrm{T}_{24}=\left[-\alpha q_7\cos Qt-q_8\sin Q\,\mathrm{t}+\frac{\alpha\mathrm{q}_1}{\mathrm{Q}^2}+\frac{\alpha\mathrm{q}_2\cos 2t}{Q^2-4}+\frac{\mathrm{q}_6\sin 2t}{Q^2-4}\right.$$
$$\left.+\alpha\mathrm{q}_3\left\{\frac{\cos(Q-1)\,\mathrm{t}}{2\mathrm{Q}-1}-\frac{\cos(Q+1)\,\mathrm{t}}{2\mathrm{Q}+1}\right\}-\frac{q_4\sin(Q+1)\,\mathrm{t}}{2\mathrm{Q}+1}+\frac{q_5\sin(Q-1)\,\mathrm{t}}{2\mathrm{Q}-1}\right]\Bigg|_{(\hat{\mathrm{t}}_0,0)} \qquad (4.25e)$$

Using

$$A^{21}(\hat{t}_0,0)=\left(PQ^2\right)^2T_{25}\,,\ \mathrm{T}_{25}=\left[-q_0\cos Qt+\frac{1}{2}\left\{\begin{matrix}\left\{\dfrac{1}{Q^2-1}\left\{\dfrac{\cos(Q+1)\,\mathrm{t}}{2\mathrm{Q}+1}-\dfrac{\cos(Q-1)\,\mathrm{t}}{2\mathrm{Q}-1}\right\}\right.\\ \left.+\dfrac{1}{2}\left\{\dfrac{1}{Q^2}+\dfrac{\cos 2t}{Q^2-4}\right\}\right\}\end{matrix}\right\}\right]_{(\hat{\mathrm{t}}_0,0)} \qquad (4.25f)$$

some of (4.25a-f), we have the following from the second and third of (4.23)

$$\hat{t}_{01}=-\left(\frac{A^{12}_{,\mathrm{t}}+A^{11}_{,\tau}}{A^{11}_{,\mathrm{t\,t}}}\right)\Bigg|_{(\hat{\mathrm{t}}_0,0)}=-\frac{1}{T_{20}}\left(T_{21}+\frac{\delta_1\cos Q\,\hat{\mathrm{t}}_0}{Q^2-1}\right);\ \hat{\mathrm{t}}_{10}=-\left(\frac{A^{21}_{,\mathrm{t}}}{A^{11}_{,\mathrm{t\,t}}}\right)\Bigg|_{(\hat{\mathrm{t}}_0,0)}=-\frac{T_{22}}{T_{20}} \qquad (4.26)$$

The maximum displacement $\xi_{1\mathrm{M}}$ of ξ_1 in (4.2) is now evaluated from (4.20), using (4.22), (4.24b) and (4.26) to get

$$\xi_{1\mathrm{M}} = \in \left[\bar{\xi} \mathrm{A}^{11} + \bar{\xi}^2 \left(\hat{t}_0 A^{11}_{,\tau} + A^{12} \right) \right] \Big|_{(\hat{t}_0,0)}$$
$$+ \in^2 \left[\bar{\xi} \mathrm{A}^{21} + \bar{\xi}^2 \left(\hat{t}_{10} A^{11}_{,\tau} + \hat{t}_{10} A^{12}_{,t} + \hat{t}_{01} A^{21}_{,t} + \hat{t}_0 A^{21}_{,\tau} + A^{22} \right) \right] \Big|_{(\hat{t}_0,0)} + \cdots \qquad (4.27)$$

On substituting into (4.27), we get

$$\xi_{1\mathrm{M}} = \in C_3 + \in^2 C_4 + \cdots \qquad (4.28a)$$

$$C_3 = PQ^2 C_5, \mathrm{C}_5 = \left[\frac{\bar{\xi}}{Q^2 - 1} \left(\cos t - CosQ\ \mathrm{t} \right) + \delta_1 \bar{\xi}^2 \left\{ \frac{t \cos Q\ \mathrm{t}}{\mathrm{Q}^2 - 1} + \frac{\left(Q^2 + 1\right)}{\left(Q^2 - 1\right)} \left(2 \sin \mathrm{t} - \sin \mathrm{Q\ t} \right) \right\} \right] \Bigg|_{(\hat{t}_0,0)} \qquad (4.28b)$$

$$C_4 = \left(PQ^2\right)^2 C_6\ , \mathrm{C}_6 = \left[\bar{\xi} T_{25} + \bar{\xi}^2 \left\{ \frac{\hat{t}_{10} \delta_1 \cos Q\ \hat{\mathrm{t}}_0}{\mathrm{PQ}^2 \left(Q^2 - 1\right)} + \frac{\hat{t}_{10} T_{21}}{PQ^2} + \hat{t}_{01} T_{22} + \hat{t}_0 T_{23} + T_{24} \right\} \right] \qquad (4.28c)$$

Meanwhile, from the approximation in (4.1), we have

$$\xi_0(t) \cong \in \left(P \cos t + \frac{2K_0}{\lambda_C} A^{11} \bar{\xi}^2 \right) + \frac{K_0 \in^2 \bar{\xi}^2}{\lambda_C} \left\{ \left(A^{11}\right)^2 + 2A^{21} \right\} + O\left(\in \bar{\xi}^3\right) + O\left(\in^2 \bar{\xi}^3\right) \qquad (4.29)$$

The condition for the maximum, $\xi_{0\mathrm{M}}$, of $\xi_0(t)$ is

$$\xi_{0,\mathrm{t}}\left(\hat{t}_a, \hat{\tau}_a\right) + \bar{\xi} \xi_{0,\tau}\left(\hat{t}_a, \hat{\tau}_a\right) = 0 \qquad (4.30a)$$

where $\hat{t}_a$ and $\hat{\tau}_a$ are the respective values of t and τ for $\xi_{0\mathrm{M}}$ to be attained. We now let

$$\hat{t}_a = \hat{T}_0 + \bar{\xi} \hat{T}_{01} + \in \left(\hat{T}_{10} + \bar{\xi} T_{11} \right) + \cdots ; \hat{\tau}_{\mathrm{a}} = \bar{\xi}\ \hat{t}_a = \bar{\xi} \left\{ \hat{T}_0 + \bar{\xi} \hat{T}_{01} + \in \left(\hat{T}_{10} + \bar{\xi} T_{11} \right) + \cdots \right\} \qquad (4.30b)$$

Using (4.30b), we clearly observe that eventually $\xi_{0\mathrm{M}}$ will have the form

$$\xi_{0\ \mathrm{M}} = P \in \left[\cos t + \frac{K_0 \in^2 \bar{\xi}^2}{\lambda_C} \right]_{(\hat{T}_0,0)} + \in^2 \frac{K_0 \in^2 \bar{\xi}^2}{\lambda_C} \left[\left(A^{11}\right)^2 + 2A^{21} \right]_{(\hat{T}_0,0)} \qquad (4.31)$$

Thus, From (4.31), we need to determine only $\hat{T}_0$ which we now perform from the equation

$$A^{11}_{,\mathrm{t}}\left(\hat{T}_0, 0\right) = 0 \qquad (4.32a)$$

This yields the value

$$\hat{T}_0 = \pi \qquad (4.32b)$$

where we have taken only the least nontrivial value of $\hat{T}_0$ in (4.32b). Thus ,from (4.31), we have

$$\xi_{0\mathrm{M}} = \in C_7 + \in^2 C_8\ ;\ \mathrm{C}_7 = PQ^2 C_9\ \ ,\ \mathrm{C}_8 = \left(PQ^2\right)^2 C_{10} \qquad (4.33a)$$

$$C_9 = -\left\{ \frac{1}{Q^2} + \frac{2K_0 \bar{\xi}^2 \left(1 + \cos Q\ \hat{\mathrm{T}}_0\right)}{\lambda_C \left(Q^2 - 1\right)} \right\}\ ; \mathrm{C}_{10}\ = \bar{\xi}^2 \left\{ 2T_{25} - \frac{\left(1 + \cos Q\ \hat{\mathrm{T}}_0\right)}{\left(Q^2 - 1\right)^2} \right\} \qquad (4.33b)$$

The net maximum displacement ξ_M is $\xi_M = \xi_{0\mathrm{M}} + \xi_{1\mathrm{M}}$ and this eventually gives

$$\xi_M = \in C_{11} + \in^2 C_{12} + \cdots ; \mathrm{C}_{11} = \left(C_3 + C_7\right)\ ;\ \mathrm{C}_{12} = \left(C_4 + C_8\right) \qquad (4.34)$$

It is now obvious that equation (4.34a) is similar to (3.63a) so that the maximum displacement $\xi_M\left(\lambda_D\right)$ at buckling is

$$\xi_M\left(\lambda_D\right) = \frac{C_{11}^2}{2C_{12}} \qquad (4.35)$$

As in equation (3.65), the dynamic buckling load λ_D follows immediately to give

$$\lambda_D\left(\frac{\omega_1}{\omega_0}\right)^2=\frac{C_{11}}{4C_{12}}=\frac{(C_5+C_9)}{4(C_6+C_{10})},\ \text{where, PQ}^2=1 \qquad (4.36)$$

5.0 Analysis of results

The result (3.66) is valid in the intervals $0<\theta<1,\ 1<\theta<2,\ \text{etc}$. and is also valid for $Q\neq\frac{1}{2},1,2,\theta,2\theta,1+\theta,$ and $(1\text{-}\theta)$ etc. It is also valid provided $0<\delta_0<1,\ 0<\delta_1<1$. The static buckling load λ_S in this case is easily obtained as

$$(1-\lambda_S)^2=4\lambda_S\bar{\xi}\left(\alpha+\frac{K_0\bar{\xi}}{\lambda_C}\right) \qquad (5.1)$$

Thus, from (3.66) and (5.1) we have

$$\left(\frac{\lambda_D}{\lambda_S}\right)=\frac{\bar{\xi}\left(\frac{C_1}{C_2}\right)\left(\frac{\omega_0}{\omega_1}\right)^2\left(\alpha+\frac{K_0\bar{\xi}}{\lambda_C}\right)}{(1-\lambda_S)^2} \qquad (5.2)$$

Hence, given either of λ_S or λ_D, we can automatically obtain the other. Similarly the result (4.36) is valid provided, $Q\neq\frac{1}{2},1,2$ and $0<\delta_1<1$. It is related to equation (5.1) by

$$\left(\frac{\lambda_D}{\lambda_S}\right)=\frac{\bar{\xi}\left(\frac{C_7}{C_8}\right)\left(\frac{\omega_0}{\omega_1}\right)^2\left(\alpha+\frac{K_0\bar{\xi}}{\lambda_C}\right)}{(1-\lambda_S)^2} \qquad (5.3)$$

We also relate equation (5.1) to (4.36) by the equation

$$\left(\frac{\lambda_D}{\lambda_S}\right)=\frac{\bar{\xi}\left(\frac{C_5+C_9}{C_6+C_{10}}\right)\left(\frac{\omega_0}{\omega_1}\right)^2\left(\alpha+\frac{K_0\bar{\xi}}{\lambda_C}\right)}{(1-\lambda_S)^2} \qquad (5.4)$$

All the results are strictly asymptotic and are generally valid for small values of the perturbation parameters $\in$ and $\bar{\xi}$.

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