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On a randomly imperfect spherical cap pressurized by a random dynamic load

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Abstract

In this paper, we investigate a dynamical system in a random setting of dual randomness in space and time variables in which both the imperfection of the structure and the load function are considered random , each with a statistical zero-mean .The auto- covariance of the load is correlated as an exponentially decaying function of the time variable .For simplicity, the normal displacement at a point on the shell surface is discretized into a symmetric pre-buckling mode and a buckling mode that has both axisymmetric and non-axisymmetric components. The imperfection is assumed in the shape of the buckling mode with its axisymmetric and non-axisymmetric amplitudes considered random-all with known first and second statistical moments. All these random parameters induce some form of randomness on the normal displacement whose mean square we shall first seek as a suitable statistical characterization of the random process for determining the dynamic buckling load which is determined asymptotically using perturbation methods.

1.0 Introduction

It has been observed that most of the existing investigations so far on the stability of elastic structures in dynamical systems have primarily concentrated on deterministic loading histories which include step loading, impulsive loading, rectangular loading, triangular loading, slowly varying time-dependent loading and some other specific loadings such as those treated by Svalbonas and Kalnins [1] , among others. Very little, in our judgment, has been recorded on extending the frontiers of investigations to the cases where the load or forcing function in a dynamic buckling setting, is random – a practical problem in man's daily life. Such practical time-dependent random loadings are many and varied and include earthquake ground motion, wind gust, the launch phase of missile flight and pyrotechnic firing-all which are treated as randomly non-stationary processes [2].

The analysis reported here however introduces randomness on two prongs of attack, first; it is introduced through the time-dependent loading history which is correlated as a slowly decaying normal "Gaussian" function of the univariate time variable. Second, it is further introduced by way of spatial coordinates through the imperfection parameters whose various amplitudes, both axisymmetric and non-axisymmetric, are considered random. In other word, randomness is here introduced in both space and time parameters.

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The study reported here is an extension of that in [3] on the same structure. However, the analysis recorded here greatly differs from that in [3] on two fronts namely (a) the admission of pre-buckling inertia and (b) the admission of randomness on the imperfection-the two which were not treated in the earlier consideration. Such inclusions significantly alter the mathematical sophistication of the present analysis.

2.0 Mathematical formulation

The original equations were derived by Danielson [4] who discretized the normal displacement $W(x,t)$ at a point on the shell surface in the form

$$W(x,y,t)=\xi_0(\tilde{t})\mathrm{W}_0(x,y)+\xi_1(\tilde{t})\mathrm{W}_1(x,y)+\xi_2(\tilde{t})\mathrm{W}_2(x,y) \qquad (2.1)$$

where W_0 , W_1 and W_2 are the symmetric pre-buckling mode, the axisymmetric buckling mode and the non-axisymmetric buckling mode respectively (all functions of spatial coordinates), and $\xi_0(\tilde{t}),\xi_1(\tilde{t})$ and, $\xi_2(\tilde{t})$ are their respective time dependent amplitudes, where $\tilde{t}$ is the time variable. He equally discretized the imperfection $\overline{W}(x,y)$ in the shape of the buckling mode, namely

$$\overline{W}(x,y)=\bar{\xi}_1 W_1(x,y)+\bar{\xi}_2 W_2(x,y) \qquad (2.2)$$

where $\bar{\xi}_1$ and $\bar{\xi}_2$ are their respective axisymmetric and non-axisymmetric amplitudes. In our study, both $\bar{\xi}_1$ and $\bar{\xi}_2$ are here considered random parameters with certain imbued statistical characterizations such as statistical means and second statistical moments, both which are considered simultaneously non-vanishing .By introducing equations (2.1) and (2.2) into the governing compatibility equation and equation of dynamic equilibrium, Danielson obtained and solved the following coupled equations , which we have here refined by the addition of uniformly viscous damping terms on all the modes, and this damping is taken proportional to the first degree of the velocity

$$\frac{1}{\omega_0^1}\frac{d^2\xi_0}{d\tilde{t}^2}+c_0\frac{d\xi_0}{d\tilde{t}}+\xi_0=\lambda F(\tilde{t}) \qquad (2.3)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{d\tilde{t}^2}+c_0\frac{d\xi_1}{d\tilde{t}}+\xi_1(1-\xi_0)-k_1\xi_1^2+k_2\xi_2^2=\bar{\xi}_1\xi_0 \qquad (2.4)$$

Here,

$$\frac{1}{\omega_0^2}\frac{d^2\xi_2}{d\tilde{t}^2}+c_0\frac{d\xi_2}{d\tilde{t}}+\xi_2(1-\xi_0)+\xi_1\xi_2=\bar{\xi}_2\xi_0 \qquad (2,5)$$

$$\xi_i(0)=\frac{d\xi_i(0)}{d\tilde{t}}=0\ ,\mathrm{i}=0,1,2 \qquad (2.6)$$

$F(\tilde{t})$ is the forcing function or loading history which, in Danielson's consideration, was the step load characterized by $F(\tilde{t})$=1 , while λ is the load amplitude , nondimensionalized with respect to the classical buckling load λ_C .Thus , we have $0<\lambda<\lambda_C\le 1$, and c_0 is the damping coefficient such that $0<c_0<<1$.Similarly, ω_i , $i=0,1,2$ are the circular frequencies of the associated modes and are such that $\frac{\omega_j}{\omega_{j-1}}<1$, $\mathrm{j}=1,2$ .We note that $k_1>0$, $\mathrm{k}_2>0$ are small constants (compared to unity) and in our quest for solution, we are to determine a particular value of λ, namely λ_D, called the dynamic buckling load , satisfying the inequality $0<\lambda_D<\lambda_S<\lambda_C\le 1$,for which the structure buckles dynamically and where λ_S is the associated static buckling load. We define λ_D as the largest load parameter for which the solution of the problem (2.3)-(2.6) remains bounded for all time $\tilde{t} > 0$ .The determination of λ_D easily follows [3, 5-8] from the maximization

$$\frac{d\lambda}{d\nabla_a^2} = 0 \qquad (2.7)$$

where ∇_a^2 is the maximum mean square displacement as a function of time variable and considered as a suitable statistical characterization of the randomly induced normal displacement .In his solution , Danielson, among other assumptions, neglected both $\bar{\xi}_1$ and $k_1\xi_1^2$. However, Ette [7, 8] has shown that some of Danielson's assumptions were either superfluous or, they over simplified the problem. Hence , we shall retain all the terms in equations (2.3)-(2.6) .In order to use the maximization (2.7), we shall first determine, using perturbation methods, a uniformly valid asymptotic expression of the normal displacement $W(\tilde{t})$ as a function of time . We note, ab initio, that by the simplification that yielded equations (2.3)-(2.6), the spatial variables have been explicitly eliminated. We shall next determine the maximum mean square normal displacement ∇_a^2,bearing in mind that the mean displacement $\tilde{W}$ is obtained from $\tilde{W} = \langle \langle \mathrm{W} \rangle \rangle$, where the averaging process $\langle\langle \cdots \rangle\rangle$ is defined as

$$\langle\langle \cdots \rangle\rangle = E[\cdots]\langle \cdots \rangle, \; \langle \cdots \rangle \equiv Lim\,\mathrm{T} \to \infty \frac{1}{\mathrm{T}}\int_0^T (\cdots)\, d\tilde{t} \qquad (2.8)$$

and $E[\cdots]$ denotes the Mathematical expectation of the quantity in square bracket $[\cdots]$ .Thus ,depending on the parameter inside the bracket $[\cdots]$, $E[\cdots]$ shall denote either the first or second statistical moment of the random parameter Z, for example , as in $E[Z] = \tilde{\mathrm{Z}}$, $E[Z^2] = \tilde{\mathrm{Z}}^2$ .The averaging process in (2.8) is both possible and plausible because the randomness of the load function $F(\tilde{t})$ is mathematically and statistically independent of the randomness of the imperfection parameters $\bar{\xi}_1$ and $\bar{\xi}_2$.

3.0 Solution of the problem

We let $t = \omega_0 \tilde{t}$, whereby we get

$$\frac{d(\;)}{d\tilde{t}} = \omega_0 \frac{d(\;)}{dt}, \; \frac{d^2(\;)}{d\tilde{t}^2} = \omega_0^2 \frac{d^2(\;)}{dt^2} \qquad (3.1)$$

On substituting (3.1) into (2.3)-(2.6), and simplifying, we get

$$\frac{d^2\xi_0}{dt^2} + 2\delta \frac{d\xi_0}{d\tilde{t}} + \xi_0 = \lambda f(t) \qquad (3.2)$$

$$\frac{d^2\xi_1}{dt^2} + 2\delta P^2 \frac{d\xi_1}{d\tilde{t}} + \xi_1\left(P^2 - P^2\xi_0\right) + P^2\left(-k_1\xi_1^2 + k_2\xi_2^2\right) = P^2\bar{\xi}_1\xi_0 \qquad (3.3)$$

$$\frac{d^2\xi_2}{dt^2} + 2\delta Q^2 \frac{d\xi_2}{dt} + \xi_2\left(Q^2 - Q^2\xi_0\right) + Q^2\xi_1\xi_2 = Q^2\bar{\xi}_2\xi_0 \qquad (3.4)$$

$$\xi_i(0) = \frac{d\xi_i(0)}{dt} = 0 \;\; , i = 0,1,2 \qquad (3.5)$$

where $P=\left(\frac{\omega_1}{\omega_0}\right), \mathrm{Q}=\left(\frac{\omega_2}{\omega_0}\right), \mathrm{c}_0\omega_0=2\delta, 0<\mathrm{P}<1,\ 0<\mathrm{Q}<1, 0<\delta<1,\ \mathrm{f}(\mathrm{t})=\mathrm{F}\left(\frac{\mathrm{t}}{\omega_0}\right)$

Here $f(t)$ is a zero-mean Gaussian stationary random dynamic load function, with exponentially decaying autocorrelation $R_f(t)=R_0 e^{-\alpha|\mathrm{t}|}$, where $0<R_0<1,\ \ 0<\alpha<1$ .The solution of (3.2) , using (3.5) for $i=0$, is

$$\xi_0(t)=\lambda\int_0^t h(\tau)f(t-\tau)\,\mathrm{d}\tau\ ,\ \mathrm{h}(\tau)=\frac{e^{-\delta\tau}}{\varphi}\sin\varphi\tau,\quad \varphi=\sqrt{(1-\delta^2)} \tag{3.6}$$

On substituting (3.6) into (3.3) for $\xi_0(t)$, we have

$$\frac{d^2\xi_1}{dt^2}+2\delta P^2\frac{d\xi_1}{d\tilde{t}}+\xi_1\left(P^2-\in G(t)\right)+P^2\left(-k_1\xi_1^2+k_2\xi_2^2\right)=\in\bar{\xi}_1 G(t) \tag{3.7}$$

$$\frac{d^2\xi_2}{dt^2}+2\delta Q^2\frac{d\xi_2}{dt}+\xi_2\left(Q^2-\in R^2G(t)\right)+Q^2\xi_1\xi_2=\in R^2\bar{\xi}_2 G(t) \tag{3.8}$$

We let

$$\xi_i(0)=\frac{d\xi_i(0)}{dt}=0\ ,\ \mathrm{i}=1,2 \tag{3.9}$$

$$\in=\lambda\left(\frac{\omega_1}{\omega_0}\right)^2, 0<\in<1,\ \ \mathrm{R}=\left(\frac{\omega_2}{\omega_0}\right), \mathrm{G}(\mathrm{t})=\int_0^t h(\tau)f(t-\tau)\,\mathrm{d}\tau \tag{3.10}$$

$$\xi_1(t)=\sum_{i=1}^{\infty}\eta_i(t)\in^i,\quad \xi_2(t)=\sum_{i=1}^{\infty}\zeta_i(t)\in^i \tag{3.11}$$

On substituting (3.11) into (3.7)-(3.10), we have the following equations

$$M\eta_1\equiv\frac{d^2\eta_1}{dt^2}+2\delta P^2\frac{d\eta_1}{dt}+P^2\eta_1=\bar{\xi}_1 G \tag{3.12}$$

$$M\eta_2=G\eta_1+P^2\left(k_1\eta_1^2-k_2\zeta_1^2\right) \tag{3.13}$$

$$N\zeta_1\equiv\frac{d^2\zeta_1}{dt^2}+2\delta\mathrm{Q}^2\frac{d\zeta_1}{dt}+Q^2\zeta_1=\bar{\xi}_2R^2G \tag{3.14}$$

On

$$N\zeta_2=R^2\zeta_1G-Q^2\eta_1\zeta_1 \tag{3.15}$$

$$\eta_i(0)=\frac{d\eta_i(0)}{dt}=\zeta_i(0)=\frac{d\zeta_i(0)}{dt}=0\ ,\ \mathrm{i}=1,2 \tag{3.16}$$

solving (3.12), we have

$$\eta_1(t)=\bar{\xi}_1\int_0^t q(\tau)G(t-\tau)\,\mathrm{d}\tau\ ,\ q(\tau)=\frac{e^{-\delta\mathrm{P}^2\tau}}{\psi}\sin\psi\tau\ ,\ \psi=P\sqrt{1-\left(\delta\mathrm{P}^2\right)^2}\ ,\ 0<\left(\delta\mathrm{P}^2\right)<1 \tag{3.17}$$

We next solve (3.14) and get

$$\zeta_1(t)=\bar{\xi}_2R^2\int_0^t p(\tau)G(t-\tau)\,\mathrm{d}\tau,\ p(\tau)=\frac{e^{-\delta Q^2\tau}}{\theta}\sin\theta\tau,\ \theta=\mathrm{Q}\sqrt{1-\left(\delta\mathrm{Q}^2\right)^2}\ ,\ 0<\left(\delta\mathrm{Q}^2\right)<1 \tag{3.18}$$

We expect the solution of (3.13), for $\eta_2(t)$, to be uniformly valid on substituting for both $\eta_1(t)$ and $\zeta_1(t)$. Thus, on solving (3.13), we have

$$\eta_2(t)=\int_0^t q(\tau)G(t-\tau)\eta_1(t-\tau)\,\mathrm{d}\tau+\mathrm{P}^2\left[k_1\int_0^t q(\tau)\eta_1^2(t-\tau)d\tau-k_2\int_0^t q(\tau)\zeta_1^2(t-\tau)\,d\tau\right] \qquad (3.19)$$

On substituting for $\eta_1(t)$ and $\zeta_1(t)$ in (3.19) from (3.15) and (3.18) and simplifying, we have

$$\eta_2(t)=\eta_2^{(1)}(t)+\eta_2^{(2)}(t)+\eta_2^{(3)}(t) \qquad (3.20a)$$

where

$$\eta_2^{(1)}(t)=\bar{\xi}_1\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_2)} q_1q_2h_3h_4\,f(t-\tau_1-\tau_3)f(t-\tau_1-\tau_2-\tau_4)d_1d_2d_3d_4 \qquad (3.20b)$$

$$\eta_2^{(2)}(t)=P^2k_1\bar{\xi}_1^2\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t_1-\tau_1-\tau_2)}\int_0^{(t_1-\tau_1-\tau_2)} q_1q_2q_3h_4h_5\,f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)d_1\cdots d_5 \quad (3.20c)$$

$$\eta_2^{(3)}(t)=-P^2k_2\left(\bar{\xi}_2R^2\right)^2\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t_1-\tau_1-\tau_2)}\int_0^{(t_1-\tau_1-\tau_2)} q_1p_2p_3h_4h_5\,f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)\times d_1\cdots d_5 \qquad (3.20d)$$

where

$h_{\mathrm{i}}=h(\tau_i),\ \mathrm{p}_{\mathrm{i}}=p(\tau_i),\mathrm{q}_{\mathrm{i}}=q(\tau_i),\mathrm{d}_{\mathrm{i}}=d(\tau_i),\mathrm{i}=1,2,3,\cdots$

The solution of (3.15) is

$$\zeta_2(t)=R^2\int_0^t p(\tau)\zeta_1(t-\tau)G(t-\tau)-Q^2\int_0^t p(\tau)\eta_1(t-\tau)\zeta_1(t-\tau)\,\mathrm{d}\tau \qquad (3.21)$$

On further simplifying equation (3.21), we have

$$\zeta_2(t)=\zeta_2^{(1)}(t)+\zeta_2^{(2)}(t) \qquad (3.22a)$$

$$\zeta_2^{(1)}(t)=R^4\bar{\xi}_2\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_2)} p_1p_2h_3h_4\,f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3)d_1\cdots d_4$$

$$\zeta_2^{(2)}(t)=-Q^2R^2\bar{\xi}_1\bar{\xi}_2\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_2)}\int_0^{(t-\tau_1-\tau_3)} p_1q_2p_3h_4h_5\,f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)d_1\cdots d_5 \quad (3.22b)$$

It follows that

$$w(t)=\xi_1(t)+\xi_2(t)=\in(\eta_1+\zeta_1)+\in^2\left(\eta_2^{(1)}+\eta_2^{(2)}+\eta_2^{(3)}+\zeta_2^{(1)}+\zeta_2^{(2)}\right)+\cdots \qquad (3.23)$$

We shall next determine the mean normal displacement $\tilde{W}=\langle\langle W(x,t)\rangle\rangle$ and now note that (i) the averaging process , namely $\langle\langle\cdots\rangle\rangle$, is distributive over individual terms in (3.23) and (ii) the upper limit of the averaging process with respect to time is hereafter evaluated at infinity ,using (2.8). We recall that

$$R_f(\tau)=R_0e^{-\alpha|\tau|}=\langle\,\mathrm{f(t)}f(t+\tau)\,\rangle,0<\alpha<1\ ,\mathrm{t}\geq 0 \qquad (3.24)$$

so that at $\tau=0$, we have $R_f(0)=R_0=\left\langle(f(t))^2\right\rangle$, which is the variance of the random load function. The zero–mean imperfection is characterized by $E[\bar{\xi}_1]=E[\bar{\xi}_2]=0$. However , non-vanishing statistical second order moments are characterized by $\mathrm{E}[\bar{\xi}_1^{\,2}]=r_1$, $\mathrm{E}[\bar{\xi}_2^{\,2}]=r_2$, $0<\mathrm{r}_1,r_2<1$, while the zero-mean load function is characterized by $\langle f(t\rangle=0$ .On executing a term-by-term evaluation of $\langle\langle w\rangle\rangle$, using (3.23), we have , as the first term, the term $\langle\langle\eta_1\rangle\rangle$ and second term $\langle\langle\zeta_1\rangle\rangle$ simplified respectively to give

$$\langle\langle\eta_1\rangle\rangle=E[\bar{\xi}_1]\int_0^\infty\int_0^\infty q_1h_2\langle f(t-\tau_1-\tau_2)\rangle d_1d_2=0,\langle\langle\zeta_1\rangle\rangle=R^2E[\bar{\xi}_2]\int_0^\infty\int_0^\infty p_1h_2\langle f(t-\tau_1-\tau_2)\rangle d_1d_2=0, \qquad (3.25a)$$

The actual evaluation of terms in $\langle\langle w\rangle\rangle$ shows that every other term vanishes except $\langle\langle\eta_2^{(2)}\rangle\rangle$ and $\langle\langle\eta_2^{(3)}\rangle\rangle$ whose values we now evaluate as follows:

$$\langle\langle\eta_2^{(2)}\rangle\rangle = P^2k_1E[\bar{\xi}_1^2]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{five\ \text{integrals}} q_1q_2q_3h_4h_5\langle f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)\rangle \mathrm{d}_1\cdots d_5$$

$$= P^2k_1\underbrace{\int_0^\infty\cdots\int_0^\infty}_{five\ \text{integrals}} q_1q_2q_3h_4h_5R_f(\tau_2+\tau_4-\tau_3-\tau_5)\mathrm{d}_1\cdots d_5 = P^2k_1r_1R_0T_2T_4T_5T_6T_7 \qquad (3.25b)$$

where

$$T_2=\int_0^\infty q(\tau)d\tau=\frac{1}{(\delta P^2)^2+\psi^2},\ T_4=\int_0^\infty h(\tau)e^{-\alpha\tau}d\tau=\frac{1}{(\delta+\alpha)^2+\varphi^2} \qquad (3.25c)$$

$$T_5=\int_0^\infty q(\tau)e^{\alpha\tau}d\tau=\frac{1}{(\delta P^2-\alpha)^2+\psi^2},\alpha<\delta P^2,\ T_6=\int_0^\infty h(\tau)e^{\alpha\tau}d\tau=\frac{1}{(\delta-\alpha)^2+\varphi^2},\alpha<\delta \quad (3.25d)$$

$$T_7=\int_0^\infty q(\tau)e^{-\alpha\tau}d\tau=\frac{1}{(\delta P^2+\alpha)^2+\psi^2} \qquad (3.25e)$$

We also have

$$\langle\langle\eta_2^{(3)}\rangle\rangle = -R^2R^4k_2E[\bar{\xi}_2^2]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{five integrals}} q_1p_2p_3h_4h_5\langle f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)\rangle d_1\cdots d_5$$

$$= -P^2R^4k_2r_2\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{five integrals}} q_1p_2p_3h_4h_5R_f(\tau_2+\tau_4-\tau_3-\tau_5)d_1\cdots d_5 = -P^2R^4R_0k_2r_2T_2T_4T_6T_8T_9 \quad (3.26a)$$

It

where

$$T_8=\int_0^\infty p(\tau)e^{\alpha\tau}d\tau=\frac{1}{(\delta Q^2-\alpha)^2+\theta^2},\alpha<\delta Q^2,\ T_9=\int_0^\infty p(\tau)e^{-\alpha\tau}d\tau=\frac{1}{(\delta Q^2+\alpha)^2+\theta^2}, \quad (3.26b)$$

follows that

$$\langle\langle w(t)\rangle\rangle = R_0P^2\in^2\left[k_1r_1T_2T_4T_5T_6T_7-k_2R^4r_2T_2T_4T_6T_8T_9\right] \qquad (3.27)$$

In the above evaluation, we have used the assumption that $E[\bar{\xi}_1\bar{\xi}_2]=E[\bar{\xi}_1]\mathrm{E}[\bar{\xi}_2]$,which implies that $\bar{\xi}_1$ and $\bar{\xi}_2$ are statistically independent.

3.1 Maximum square displacement

The autocorrelation $R_w(t_1,t_2)$ of the displacement $w(t)$ is defined as

$$R_w(t_1,t_2)=\left\langle\left\langle\ \{w(t_1)-\langle\langle w(t_1)\rangle\rangle\ \}\{w(t_2)-\langle\langle w(t_2)\rangle\rangle\ \}\right\rangle\right\rangle \tag{3.28a}$$

It is also defined as

$$R_w(\tau)=\left\langle\left\langle\ \{w(t)-\langle\langle w(t)\rangle\rangle\ \}\{w(t+\tau)-\langle\langle w(t+\tau)\rangle\rangle\ \}\right\rangle\right\rangle \tag{3.28b}$$

Thus when $\tau=0$, we have the mean square displacement $\nabla^2(t)$ as

$$\nabla^2(t)=R_w(0)=\left\langle\left\langle\{w(t)\}^2\right\rangle\right\rangle-\langle\langle w(t)\rangle\rangle^2 \tag{3.29}$$

In determining $\nabla^2(t)$, we substitute into (3.29) and note that we only need to evaluate $\left\langle\left\langle\ \{w(t)\}^2\right\rangle\right\rangle$ because the second term , namely $\langle\langle w(t)\rangle\rangle^2$, is obtainable from (3.27). Using (3.23), we obtain

$$\begin{aligned}\left\langle\left\langle\{w(t)\}^2\right\rangle\right\rangle=&\Big\langle\Big\langle\in^2\left(\eta_1^2+2\eta_1\zeta_1+\zeta_1^2\right)+2\in^3\left(\eta_1\eta_2^{(1)}+\eta_1\eta_2^{(2)}+\eta_1\eta_2^{(3)}+\eta_1\zeta_2^{(1)}+\eta_1\zeta_2^{(2)}\right.\\&\left.+\zeta_1\eta_2^{(1)}+\zeta_1\eta_2^{(2)}+\zeta_1\eta_2^{(3)}+\zeta_1\zeta_2^{(1)}+\zeta_1\zeta_2^{(2)}\right)+\in^4\Big\{\Big\{\eta_2^{(1)\,2}+\eta_2^{(2)\,2}+\eta_2^{(3)\,2}+\zeta_2^{(1)\,2}+\zeta_2^{(2)\,2}+\\&+2\Big\{\eta_2^{(1)}\left(\eta_2^{(2)}+\eta_2^{(3)}+\zeta_2^{(1)}+\zeta_2^{(2)}\right)+\eta_2^{(2)}\left(\eta_2^{(3)}+\zeta_2^{(1)}+\zeta_2^{(2)}\right)+\eta_2^{(3)}\left(\zeta_2^{(1)}+\zeta_2^{(2)}\right)+\zeta_2^{(1)}\zeta_2^{(2)}\Big\}\Big\}\Big\rangle\Big\rangle+\cdots\end{aligned} \tag{3.30}$$

We now evaluate each term in (3.30).By omitting tentatively the multiplicative factors $\in^i$, $i=2,3,4$ in the terms in (3.30), we have

$$\begin{aligned}\left\langle\left\langle\eta_1^2\right\rangle\right\rangle&=E\,[\bar{\xi}_1^2]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{four integrals}}q_1q_2h_3h_4\langle f(t-\tau_1-\tau_3)f(t-\tau_2-\tau_4)\rangle\,\mathrm{d}_1\cdots d_4\\&=r_1\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{four integrals}}q_1q_2h_3h_4R_f(\tau_1+\tau_3-\tau_2-\tau_4)\,\mathrm{d}_1\cdots d_4=r_1R_0T_4T_6^2T_7\end{aligned} \tag{3.31}$$

$$2\langle\langle\eta_1\zeta_2\rangle\rangle=2E\,[\bar{\xi}_1]E\,[\bar{\xi}_2]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{four integrals}}q_1p_2h_3h_4R_f(\tau_1+\tau_3-\tau_2-\tau_4)=0 \tag{3.32}$$

$$\left\langle\left\langle\zeta_1^2\right\rangle\right\rangle=E\,[\bar{\xi}_2^2]\,\mathrm{R}^4\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{four integrals}}p_1p_2h_3h_4R_f(\tau_1+\tau_3-\tau_2-\tau_4)\,\mathrm{d}_1\cdots d_4=r_2R_0R^4T_4T_6T_8T_9 \tag{3.33}$$

We note that virtually all the terms of order $\in^3$ in (3.30) vanish on evaluation. A typical term of this order is

$$2\langle\langle\eta_1\eta_2\rangle\rangle=2\mathrm{E}[\bar{\xi}_1^2]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{six integrals}}q_1h_2q_3q_4h_5h_6\langle f(t-\tau_1-\tau_2)f(t-\tau_3-\tau_6)f(t-\tau_3-\tau_4-\tau_6)\rangle\,\mathrm{d}_1\cdots d_6 \tag{3.34a}$$

To evaluate this and other similar terms of this order, we note, as in [9, page 343, equations (7.28) – (7.29b)], that, if $Z_j = f_j - \langle f_j \rangle$, where $f_j = f(t_j)$, then

$$\langle Z_1 Z_2 \cdots Z_{2m} \rangle = \sum_{all\ \text{pairs}} \left(\prod \langle Z_j Z_k \rangle \right) = \sum_{all\ \text{pairs}} \left(\langle Z_j Z_k \rangle \langle Z_l Z_p \rangle \cdots \langle Z_q Z_s \rangle \right) \qquad (3.34b)$$

for $j \neq k \neq l \neq p$, etc., and

$$\langle Z_1 Z_2 \cdots Z_{2m+1} \rangle = 0 \qquad (3.34c)$$

This means that, in (3.34b), the number of averages over pairs is equal to the number of different ways that $2m$ different variables in

$Z_1 Z_2 \cdots Z_{2m}$ can be chosen in pairs. Of course, this number of ways is equal to $\frac{(2m)!}{2^m m!}$ .Thus if, for example, $m = 2$ (as in our case), then, there are three different ways in which the product $\langle Z_1 Z_2 Z_3 Z_4 \rangle$ factors out into the following three co-variances

$$\langle Z_1 Z_2 Z_3 Z_4 \rangle = \langle Z_1 Z_2 \rangle \langle Z_3 Z_4 \rangle + \langle Z_2 Z_3 \rangle \langle Z_1 Z_4 \rangle + \langle Z_1 Z_3 \rangle \langle Z_2 Z_4 \rangle \qquad (3.34d)$$

However ,if m is odd, (for example m =1), then from (3.34c), we have a product of three values, namely $\langle Z_1 Z_2 Z_3 \rangle$,to be chosen in pairs and we thus have

$$\langle Z_1 Z_2 Z_3 \rangle = 0 \qquad (3.34e)$$

On applying (3.34e) to (3.34a), we observe that in this case, $m = 1$ and $Z_1 = f(t - \tau_1 - \tau_2)$, $Z_2 = f(t - \tau_3 - \tau_6)$, $Z_3 = f(t - \tau_3 - \tau_4 - \tau_6)$ and therefore we have $\langle f(t - \tau_1 - \tau_2) f(t - \tau_3 - \tau_6) f(t - \tau_3 - \tau_4 - \tau_6) \rangle = 0$. Therefore we have $2\langle\langle \eta_1 \eta_2 \rangle\rangle = 0$. Alternatively, we can write the following

$$\begin{aligned} &\langle f(t - \tau_1 - \tau_2) f(t - \tau_3 - \tau_6) f(t - \tau_3 - \tau_4 - \tau_6) \rangle = \\ &\qquad \langle f(t - \tau_1 - \tau_2) f(t - \tau_3 - \tau_6) \rangle \langle f(t - \tau_3 - \tau_4 - \tau_6) \rangle \rangle = 0 \end{aligned} \qquad (3.35)$$

where (3.35) vanishes because $\langle f(t - \tau_3 - \tau_4 - \tau_6) \rangle \rangle = 0$. We now simplify terms of order $\in^4$ in (3.30) and the first term $\langle\langle \eta_2^{(1)\,2} \rangle\rangle$ simplifies to

$$\begin{aligned} \langle\langle \eta_2^{(1)\,2} \rangle\rangle = E[\bar{\xi}_1^2] \underbrace{\int_0^\infty \cdots \int_0^\infty}_{eight\ \text{integrals}} & q_1 q_2 h_3 h_4 q_5 q_6 h_7 h_8 \langle f(t - \tau_1 - \tau_3) f(t - \tau_1 - \tau_2 - \tau_4) \times \\ & f(t - \tau_5 - \tau_7) f(t - \tau_5 - \tau_6 - \tau_8) \mathrm{d}_1 \cdots d_8 \end{aligned} \qquad (3.36a)$$

We easily recognize (3.36a) as a four-point correlation whereby $m = 2$ (as in (44d)).Thus, as expected, (3.36a) factors out into the following three covariances

$$\begin{aligned} \langle\langle \eta_2^{(1)\,2} \rangle\rangle = E[\bar{\xi}_1^2] \underbrace{\int_0^\infty \cdots \int_0^\infty}_{eight\text{integrals}} & q_1 q_2 h_3 h_4 q_5 q_6 h_7 h_8 \big[R_f(\tau_3 - \tau_2 - \tau_4) R_f(\tau_7 - \tau_6 - \tau_8) + R_f(\tau_1 + \tau_3 - \tau_5 - \tau_7) \times \\ & R_f(\tau_1 + \tau_2 + \tau_4 - \tau_5 - \tau_6 - \tau_7) + R_f(\tau_1 + \tau_3 - \tau_5 - \tau_6 - \tau_8) R_f(\tau_1 + \tau_2 + \tau_4 - \tau_5 - \tau_7) \big] \mathrm{d}_1 \cdots d_8 \end{aligned} \qquad (3.36b)$$

The first iterated integral in (3.36b) is easily performed to give $R_0^2 r_1 T_2^2 T_4^2 T_5^2 T_6$ while the second and third integrals in (3.36b) have the same value on evaluation and each gives $R_0^2 r_1 T_4^2 T_5 T_6^2 T_7 T_{12} T_{13}$, where

$$T_{12}=\int_0^\infty qe^{-2\alpha\tau}d\tau=\frac{1}{(\delta P^2+2\alpha)^2+\psi^2}\,,\; T_{13}=\int_0^\infty qe^{2\alpha\tau}d\tau=\frac{1}{(\delta P^2-2\alpha)^2+\psi^2}\,, 2\alpha<\delta P^2 \quad (3.36c)$$

Thus, we haven $$\left\langle\left\langle \eta_2^{(1)\,2}\right\rangle\right\rangle = r_1R_0^2\left(T_2^2T_4^2T_5^2T_6 + 2T_4^2T_5T_6^2T_{6^2}T_{12}T_{13}\right) \quad (3.36d)$$

We now evaluate the term $\left\langle\left\langle \eta_2^{(2)\,2}\right\rangle\right\rangle$ as

$$\left\langle\left\langle \eta_2^{(2)\,2}\right\rangle\right\rangle = P^4k_1^2E[\bar{\xi}_1^4]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{ten integrals}} q_1q_2q_3h_4h_5q_6q_7q_8h_9h_{10}\langle f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)\times \quad (3.37a)$$
$$f(t-\tau_6-\tau_7-\tau_9)f(t-\tau_6-\tau_8-\tau_{10})\rangle\, \mathrm{d}_1\cdots d_{10}$$

$$=3k_1^2\left(E[\bar{\xi}_1^2]\right)^2\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{ten integrals}} q_1q_2q_3h_4h_5q_6q_7q_8h_9h_{10}\left[R_f(\tau_2+\tau_4-\tau_3-\tau_5)R_f(\tau_7+\tau_9-\tau_8-\tau_{10})\right.$$
$$+R_f(\tau_1+\tau_2+\tau_4-\tau_6-\tau_7-\tau_9)R_f(\tau_1+\tau_3+\tau_5-\tau_6-\tau_8-\tau_{10}) \quad (3.37b)$$
$$\left.+R_f(\tau_1+\tau_2+\tau_4-\tau_6-\tau_8-\tau_{10})R_f(\tau_1+\tau_3+\tau_5-\tau_6-\tau_7-\tau_9)\right]\mathrm{d}_1\cdots d_{10}$$

The first integral in (3.37b) has the value $3P^4k_1^2R_0^2r_1^2T_2^2T_4^2T_5^2T_6^2T_7^2$ while the second and third in (3.37b) are equal on evaluation and each of them is evaluated to give $3P^4k_1^2r_1^2R_o^2T_4^2T_5^2T_5^2T_6^2T_7^2T_{12}$. Thus we have

$$\left\langle\left\langle \eta_2^{(2)\,2}\right\rangle\right\rangle = 3P^2r_1^2k_1^2R_0^2\left[T_2^2T_4^2T_5^2T_6^2T_7^2 + 2T_4^2T_5^2T_5^2T_6^2T_7^2T_{12}\right] \quad (3.37c)$$

We next evaluate the term $\left\langle\left\langle \eta_2^{(3)\,2}\right\rangle\right\rangle$ in the following way:

$$\left\langle\left\langle \eta_2^{(3)\,2}\right\rangle\right\rangle = P^4k_2^2R^8E[\bar{\xi}_2^4]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{ten integrals}} q_1p_2p_3h_4h_5q_6p_7p_8h_9h_{10}\langle f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)\times$$
$$f(t-\tau_6-\tau_7-\tau_9)f(t-\tau_6-\tau_8-\tau_{10})\rangle\,\mathrm{d}_1\cdots d_{10} \quad (3.38a)$$

$$=3k_2^2\left(E[\bar{\xi}_2^2]\right)^2\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{ten integrals}} q_1p_2p_3h_4h_5q_6p_7p_8h_9h_{10}\left[R_f(\tau_2+\tau_4-\tau_3-\tau_5)R_f(\tau_7+\tau_9-\tau_8-\tau_{10})\right.$$
$$+R_f(\tau_1+\tau_2+\tau_4-\tau_6-\tau_7-\tau_9)R_f(\tau_1+\tau_3+\tau_5-\tau_6-\tau_8-\tau_{10})$$
$$\left.+R_f(\tau_1+\tau_2+\tau_4-\tau_6-\tau_8-\tau_{10})R_f(\tau_1+\tau_3+\tau_5-\tau_6-\tau_7-\tau_9)\right]\mathrm{d}_1\cdots d_{10} \quad (3.38b)$$

On evaluation, the first iterated integral in (3.38b) gives $3P^4k_2^2R^8r_2^2R_0^2T_4^2T_5^2T_8^2T_9^2$, while the second and third integrals in (3.38b) are equal with each evaluated as $3P^4k_2^2R^8r_2^2R_0^2T_4^2T_5^2\ \ T_8^2T_9^2T_{12}T_{13}$. Thus, we have

$$\left\langle\left\langle \eta_2^{(3)\,2}\right\rangle\right\rangle = 3P^4k_2^2R^8r_2^2R_0^2\left[T_4^2T_5^2T_8^2T_9^2 + 2T_4^2T_5^2T_8^2T_9^2T_{12}T_{13}\right] \quad (3.38c)$$

We next evaluate the term $\left\langle\left\langle \zeta_2^{(1)\,2}\right\rangle\right\rangle$ as

$$\left\langle\left\langle \zeta_2^{(1)\,2}\right\rangle\right\rangle = R^8\ \mathrm{E}\,[\bar{\xi}_2^{\,2}]\ \underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{eight integrals}} p_1p_2h_3h_4p_5p_6h_7h_8\big\langle f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3)\times \tag{3.39a}$$
$$f(t-\tau_5-\tau_6-\tau_8)f(t-\tau_5-\tau_7)\big\rangle\,\mathrm{d}_1\cdots d_8$$

$$= R^8\ \mathrm{r}_2\ \underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{eight integrals}} p_1p_2h_3h_4p_5p_6h_7h_8\Big[R_f(\tau_2+\tau_4-\tau_3)R_f(\tau_8+\tau_6-\tau_7)$$
$$+\ R_f(\tau_1+\tau_2+\tau_4-\tau_5-\tau_6-\tau_8)R_f(\tau_1+\tau_3-\tau_5-\tau_7) \tag{3.39b}$$
$$+R_f(\tau_1+\tau_2+\tau_4-\tau_5-\tau_7)R_f(\tau_1+\tau_3-\tau_5-\tau_6-\tau_8)\Big]\mathrm{d}_1\cdots d_8$$

The first term in (3.39b) has the value $R_0^2R^8r_2T_3^2T_4^2T_6^2T_9^2$

where

$$T_3 = \int_0^\infty p(\tau)d\tau = \frac{1}{(\delta\,Q^2)^2+\theta^2} \tag{3.39c}$$

While the second and terms in (3.39b) have equal value on determination, each of value $R_0^2R^8r_2T_4^2T_6^2T_8T_9T_{14}T_{15}$ where

$$T_{14} = \int_0^\infty p\,e^{2\alpha\tau}d\tau = \frac{1}{(\delta\,\mathrm{Q}^2-2\alpha)^2+\theta^2},\ 2\alpha<\delta\,\mathrm{Q}^2\,;\ T_{15} = \int_0^\infty p\,e^{-2\alpha\tau}d\tau = \frac{1}{(\delta\,\mathrm{Q}^2+2\alpha)^2+\theta^2} \tag{3.39d}$$

Thus, we have

$$\left\langle\left\langle \zeta_2^{(1)\,2}\right\rangle\right\rangle = R^8r_2R_0^2\left(T_3^2T_4^2T_6^2T_9^2+2T_4^2T_6^2T_8T_9T_{14}T_{15}\right) \tag{3.39e}$$

The term $\left\langle\left\langle \zeta_2^{(2)\,2}\right\rangle\right\rangle$ is next evaluated as

$$\left\langle\left\langle \zeta_2^{(2)\,2}\right\rangle\right\rangle = (QR)^4E[\bar{\xi}_1^2]\,\mathrm{E}[\bar{\xi}_2^2]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{ten integrals}} p_1q_2p_3h_4h_5p_6q_7p_8h_9h_{10}\big\langle f(t-\tau_1-\tau_2-\tau_4)f(t-\tau_1-\tau_3-\tau_5)\times \tag{3.40a}$$
$$f(t-\tau_6-\tau_7-\tau_9)f(t-\tau_6-\tau_8-\tau_{10})\big\rangle\,\mathrm{d}_1\cdots d_{10}$$

$$=(QR)^4r_1r_2\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{ten integrals}} p_1q_2p_3h_4h_5p_6q_7p_8h_9h_{10}\Big[R_f(\tau_2+\tau_4-\tau_3-\tau_5)R_f(\tau_7+\tau_9-\tau_8-\tau_{10})$$
$$+R_f(\tau_1+\tau_2+\tau_4-\tau_6-\tau_7-\tau_9)R_f(\tau_1+\tau_3+\tau_5-\tau_6-\tau_8-\tau_{10}) \tag{3.40b}$$
$$+R_f(\tau_1+\tau_2+\tau_4-\tau_6-\tau_7-\tau_9)R_f(\tau_1+\tau_3+\tau_5-\tau_6-\tau_7-\tau_9)\Big]\mathrm{d}_1\cdots d_{10}$$

The first integral in (3.40b) is evaluated as $R_0^2(QR)^4r_1r_2T_3^2T_4^2T_6^2T_7^2T_8^2$, while the second and third integrals in (3.40b) are equal on evaluation, each of value $R_0^2(QR)^4r_1r_2T_3^2T_4^2\ \ T_6^2T_7^2T_8T_9T_{14}T_{15}$. Thus we have

$$\left\langle\left\langle \zeta_2^{(2)\,2}\right\rangle\right\rangle = R_0^2(QR)^4 r_1 r_2\left(T_3^2T_4^2T_6^2T_7^2T_8^2 + 2T_4^2T_6^2T_7^2T_8T_9T_{14}T_{15}\right) \qquad (3.40c)$$

We now display a typical term of order $\in^4$ that multiplies the number 2 inside the inner chain bracket $\{\cdots\}$ in equation (38), and show why this and all other similar terms will vanish on evaluation. One of such terms is $2\left\langle\left\langle \eta_2^{(1)}\eta_2^{(2)}\right\rangle\right\rangle$ simplified as follows

$$2\left\langle\left\langle \eta_2^{(1)}\eta_2^{(2)}\right\rangle\right\rangle = 2P^2k_1E[\bar{\xi}_1^3]\underbrace{\int_0^\infty\cdots\int_0^\infty}_{\text{nine integrals}} q_1q_2h_3h_4q_5q_6q_7h_8h_9\langle f(t-\tau_1-\tau_3)f(t-\tau_1-\tau_2-\tau_4)\times$$
$$f(t-\tau_5-\tau_6-\tau_8)f(t-\tau_5-\tau_7-\tau_9)\rangle\, \mathrm{d}_1\cdots d_9 \qquad (3.41a)$$

We know that

$$E[\bar{\xi}_1^3] = E[\bar{\xi}_1^2]\,\mathrm{E}[\bar{\xi}_1] = 0 \qquad (3.41b)$$

where (3.41b) follows because $\mathrm{E}[\bar{\xi}_1]=0$. Therefore $2\left\langle\left\langle \eta_2^{(1)}\eta_2^{(2)}\right\rangle\right\rangle$ = 0.Every other term in the inner chain bracket vanishes on similar circumstances. It therefore follows that the maximum mean square displacement ∇_a^2 now takes the form

$$\nabla_a^2 = \in^2 C_1 + \in^4 C_2 + O\left(\in^5\right) \qquad (3.42a)$$

$$C_1 = R_0C_3\,,\ \mathrm{C}_3 = \left[r_1T_4T_6^2T_7 + r_2R^4T_4T_6T_8T_9\right],\ \mathrm{C}_2 = R_0^2C_4 \qquad (3.42b)$$

$$C_4 = \Big[r_1T_2^2T_4^2T_5^2T_6 + 2T_4^2T_5T_6^2T_{12}T_{13} + 3P^4k_1^2r_1^2\left\{T_2^2T_4^2T_5^2T_6^2T_7^2 + 2T_4^2T_5^2T_6^2T_7^2T_{12}\right\}$$
$$+3P^4k_2^2R^8r_2^2\left\{T_2^2T_4^2T_6^2T_8^2T_9^2 + 2T_4^2T_5^2T_8^2T_9^2T_{12}T_{13}\right\} + R^8r_2\left\{T_3^2T_4^2T_6^2T_9^2 + 2T_4T_6^2T_8T_9T_{14}T_{15}\right\} \qquad (3.42c)$$
$$+(QR)^4r_1r_2\left\{T_3^2T_4^2T_6^2T_7^2T_8 + 2T_4^2T_6^2T_7^2T_8T_9T_{14}\right\} - \left(\frac{\langle\langle w(t)\rangle\rangle}{R_0}\right)^2\Big]$$

where $\langle\langle w(t)\rangle\rangle$ is as obtained in (35) .

3.2: Dynamic buckling load λ_D

The dynamic buckling load λ_D is obtained from the maximization (2.7), and the usual procedure is to first reverse the series (3.42a) so that we have

$$\in^2 = d_1\nabla_a^2 + d_2\left(\nabla_a^2\right)^2 + \cdots \qquad (3.43a)$$

By substituting for ∇_a^2 from (3.42a-c) into (3.43a) and equating the coefficients of $\in^2$ and $\in^4$, we get respectively

$$d_1 = \frac{1}{C_1},\ \text{ and }\ \mathrm{d}_2 = -\frac{C_2}{C_1^3} \qquad (3.43b)$$

The maximization (2.7) easily follows directly from (3.43a), to give

$$\nabla_a^2(\lambda_D) = -\frac{d_1}{2d_2} = \frac{C_1^2}{2C_2} \qquad (3.43c)$$

If we evaluate (3.43a) at $\lambda=\lambda_D$, we get $\in_D^2 = \dfrac{C_1}{4C_2}$ (3.43d) where $\in_D$ is the value of $\in$ at $\lambda = \lambda_D$.On substituting for C_1 and C_2 in (3.43d) from (3.40a,b), we have

$$\left\{ \lambda_D \left(\frac{\omega_1}{\omega_0} \right)^2 \right\}^2 = \left(\frac{C_3}{4R_0C_4} \right) \qquad (3.44)$$

4.0 Analysis of result and conclusion

Since the right hand side of equation (3.44) is independent of the load parameter λ_D, the result gives a straightforward expression for determining the dynamic buckling load λ_D. We easily observe that the result is of order $R_0^{-\frac{1}{2}}$, where R_0 is the variance of the random load

By letting $k_1 = 0$, and $k_2 = 0$ in two separate instances, we obtain the effects of the absence of each of the quadratic nonlinearities, namely $k_1\xi_1^2$ and $k_2\xi_2^2$, that appear in the formulation of the problem. We note that the effects of the coupling between the buckling modes are obtained from the terms multiplying r_1r_2 in (3.44). Thus, we readily observe that the effects of the coupling between the buckling modes is possible only if we do not neglect any of the imperfection parameters because neglecting any of the imperfection parameters automatically implies neglecting the effects of the coupling of the mode that is in the shape of the neglected imperfection, with any other mode- be it buckling mode or pre-buckling mode. We strongly stress that the result is asymptotically valid for, $0 < c < 1,\ 0 < \delta < 1\ 0 < \mathrm{R} < 1, 0 < \mathrm{Q} < 1$ and $0 < P < 1$,among other conditions .It is obvious that the dynamic buckling load λ_D, depends ,among other things, on the ratios of frequencies $\left(\frac{\omega_1}{\omega_0} \right)$ and $\left(\frac{\omega_2}{\omega_0} \right)$.

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