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A mathematical model for predicting earthquake occurrence

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Abstract

We consider the continental crust under damage. We use the observed results of microseism in many seismic stations of the world which was established to study the time series of the activities of the continental crust with a view to predicting possible time of occurrence of earthquake. We consider microseism time series model with codal waves as the main source of energy and show that it is an adoptable model for predicting earthquake occurrence.

1.0 Introduction

In the previous study Oyesanya [1] considered one of the microseism time series model –Duffing oscillator. In that study ,the following were highlighteds

i. Microseism time series are nonstationary.

ii. Microseism time series are stochastic .

iii. From the point of view of data analysis , there is strong evidence in favour of a nonlinear character of microseism time series .

The same results–nonstationarity, stochasticity and nonlinearity were also obtained for the time series generated by a Duffing oscillator Guckenheimer and Holmes [2] as well as n-well potential forced oscillator , having added, in both cases , additive noise to account for stochasticity . Hence a Duffing oscillator with noise is an adoptable model for the study of microseism time series. Thus we study the Duffing oscillator

$$\ddot{q} + \delta\dot{q} - \alpha q + \beta q^3 = \gamma\cos(\omega t) \tag{1.1}$$

Where δ is the coefficient of damping, α the proper or resonant frequency of the system in the absence of external forces, β the coefficient of nonlinearity, γ the amplitude of the external harmonic force.

In this present study, we consider the second mentioned (but not treated) model in Oyesanya [1] namely

$$p = \dot{q}$$

$$\dot{p} + \frac{\partial V(q)}{\partial q} + \delta p = N_0 \sum_{i=1}^{n} e^{\lambda(t-t_i)} \tag{1.2}$$

where $V(t) = -\alpha_0 \frac{q^2}{2} + \beta\frac{q^4}{4}$ is the classical bi-stable potential and *i* stands for each coda wave contribution proposed by Correig et al [3]. We show that it is a good model for predicting earthquake occurrence.

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Coda waves are the result of a multiple scattering process. In the course of its propagation, three different regimes can be distinguished; the ballistic regime (associated with non-scattered energy), the diffusion regime (characterized by a diffusively decay with time due to multiple scattering) and the equipartitioning regime. We understand equipartitioning regime as that regime for which the energy is separated into multiple wave packets due to multiple scattering, where initial coherent wave-fronts are broken and re-radiated as in Huygens reflections.

Seismic stations are established to study the time series of the activities of the continental crust with a view to predicting possible time of occurrence of earthquake.

In this study an attempt is made to look at the model for predicting earthquake occurrence from a different viewpoint by considering coda waves as the main source of energy in the model of microseism time series proposed by Correig et al [3].

2.0 Model description

Correig et al [4] proposed the model of microseism time series given by

$$\left.\begin{aligned}\dot{q} &= p \\ \dot{p} + \frac{\partial V(q)}{\partial q} + \delta p &= \sum_{i=1}^{2} \gamma_1 \cos(\omega_1 t) + \Gamma \varepsilon(t)\end{aligned}\right\} \tag{2.1}$$

where V is defined as $V(q) = -\alpha \frac{q^2}{2} + \beta \frac{q^4}{4}$ is the potential and $\varepsilon(t)$ is random noise assumed to be white noise, δ is the coefficient of damping, β the coefficient of nonlinearity, γ_1 the amplitudes of the two external harmonic forces of frequency ω_i and Γ the noise amplitude.

As can readily be seen, equation (2.1) constitutes a generalization of the Duffing equation, with two external harmonic forces, external noise and parametric resonance. These two external forces constitute the input of energy of the system.

However, in the equipartitioned regime these forces will be negligible. As noted in Correig et al [3], the main source of energy responsible for the excitation of the equilibrium fluctuations is provided by the presence of coda waves, in the diffuse or equipartition regime, originated by the continuous occurrence of earthquakes of different magnitude and at different places, defining an extended sources.

To take into account coda waves as the main source of energy, our model (2.1) need to be modified. We consider the model of a simple exponential relaxation process, defined as

$$\left.\begin{aligned}N(t) &= N_0 e^{-\mu}, t \geq 0 \\ N(t) &= 0, t \leq 0\end{aligned}\right\} \tag{2.2}$$

As coda waves are continuously generated, we will consider a summation of exponential processes, with the inter-event time following a Poisson distribution. We assume an initial amplitude N_0 constant for each exponential process and finally, a random phase has been added. Thus, our model becomes

$$\left.\begin{aligned}p &= \dot{q} \\ \dot{p} + \frac{\partial V(q)}{\partial q} + \delta p &= N_0 \sum_{i=1}^{n} e^{\lambda(t-t_1)}\end{aligned}\right\} \tag{2.3}$$

Where actually *V(q)* is the classical bistable potential $V(q) = -\alpha_0 \frac{q^2}{2} + \frac{\beta q^4}{4}$, and the sub-index *i* stands for each of coda wave contribution.

We see that if we suppose that a device is subject to shocks that occur in accordance with a Poisson process having rate λ. The ith shock gives rise to a damage D_i. The $D_i, i \geq 1$, are assumed to be independent and identically distributed and also to be independent of $\{N(t), t \geq 0\}$, where $N(t)$

denotes the number of shocks in [0, t]. The damage due to a shock is assumed to decrease exponentially in time. That is, if a shock has an initial damage D, then a time t later its damage is $De^{-\alpha t}$.

If we suppose that the damages are additive, then $D(t)$, the damage at t, can be expressed as

$$D(t) = \sum_{i=1}^{N(t)} D_i e^{-\alpha(t-s_i)}$$

where S_i represents the arrival time of the ith shock. We can determine the expected damage $E[D(t)]$ at t as follows: see Sheldom [5]. We see that our model takes care of the nature of microseism as observed – nonststionarity, stochasticity and nonlinearity.

We note that the stability analysis of this model will help us in predicting earthquake occurrences.

3.0 Stability analysis

In general, the notions of stability of a solution and boundedness of a solution are independent; for example, the solutions $x = t + x_0$ of $x' = 1$ are stable but not bounded. However, in the case of linear systems the two notions are equivalent as seen from the established result that

All solutions of a linear system are stable if and only if they are bounded David [6]. We now investigate the stability or otherwise of the linear part of our model.

Consider our model

$$p = q'$$

$$p' + \frac{\partial V(q)}{\partial q} + \delta p = N_0 \sum_{i=1}^{n} e^{\lambda(t-t_i)}$$

where $$V(q) = -\alpha_0 \frac{q^2}{2} + \beta \frac{q^4}{4}.$$

The above model can be written in matrix form to get

$$q' = p$$

$$p' = \alpha_0 q - \delta p - \beta q^3 + N_0 K e^{\lambda t}$$

$$where \quad \mathrm{K} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{e}^{-\lambda \mathrm{t_i}}$$

i.e $$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha_0 & -\delta \end{pmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ -\beta q^3 + N_0 K e^{\lambda t} \end{bmatrix}.$$

The linear part of our model is given by

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha_0 & -\delta \end{pmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

To determine the solutions to the linear part of our model, we have that the eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ \alpha_0 & -\delta \end{pmatrix}$ are given by

$$r(r + \delta) - \alpha_0 = 0$$

$$r^2 + \delta r - \alpha_0 = 0$$

i.e
$$r_1 = -\frac{-\delta + \sqrt{\delta^2 + 4\alpha_0}}{2}$$

and
$$r_2 = -\frac{-\delta - \sqrt{\delta^2 + 4\alpha_0}}{2}.$$

Since $\delta, \alpha_0 \succ 0$, then $\sqrt{\delta^2 + 4\alpha_0} \succ 0$ and $-\delta + \sqrt{\delta^2 + 4\alpha_0} \succ 0$. Therefore we must have that $r_1 > 0$ and $r_2 < 0$. Since λ is the rate for the Poisson process, the above result show that only coda waves with rate equal to r_1 is permissible for our model. The implication of this is that one of the solutions $x_1(t) = e^{r_1 t}$ to the linear part of our model will be unbounded, since $x_1(t) = e^{r_1 t} \rightarrow \infty$ as $t \rightarrow \infty$.

Hence since not all the solutions of the linear part of our model is bounded, then we conclude that not all solutions of the linear part of our model is stable. That is to say that some of the solutions to the linear part of our model is unstable.

Having shown that the linear part of our model is unstable. To confirm the instability of our model, we will study the nonlinear part of our model together with some theorems and definitions that will help us to solve the problem. One of the result is the Njoku and Omari [7] result which we now state

***Theorem* 3.1** Njoku and Omari [7].

Assume $\delta > 0$. Moreover, suppose that $\bar{\alpha}$ is a strict lower and $\bar{\beta}$ is a strict upper solution of the equation $\ddot{q} + \delta\dot{q} + g(t,q) = h(t)$ which satisfy $\bar{\alpha} \le \bar{\beta}$. then, the equation has at least one unstable *T*-periodic solution $\hat{S}$, with $\bar{\alpha} << \hat{S} << \bar{\beta}$ provided that the number of the *T*-periodic solutions is finite.

Our model in (2.3), can be written as

$$\ddot{q} + \delta\dot{q} - \alpha_0 q + \beta q^3 = N_0 \sum_{t=1}^{n} e^{\lambda(t-t_1)} \qquad (3.1)$$

We can easily observe by comparing the differential equation $\ddot{q} + \delta\dot{q} + g(t,q) = h(t)$ in Theorem 3.1 with our model equation (3.1) that $g(t,q) = -\alpha_0 q + \beta q^3$ and $h(t) = N_0 \sum_{t=1}^{n} e^{\lambda(t-t_1)}$.

Hence our model equation (3.1) is a special case of the differential equation in Theorem 3.1 and therefore can be used to investigate the stability or otherwise of our model if in addition our model also satisfy the remaining conditions of upper and lower solutions in Theorem 3.1. To confirm that our model satisfy these conditions, we see Theorem 3.2 below.

***Theorem* 3.2**

Consider the equation (3.1) *which can be written as*

$$\ddot{q} + \delta\dot{q} - \alpha_0 q + \beta q^3 = N_0 e^{\lambda t} \sum_{t=1}^{n} e^{\lambda t_1}$$

provided that $\lambda^2 + 3\lambda\delta - 9\alpha_0 = 0$ and $\beta A^3 = N_0 K$,

where $$K = \sum_{i=1}^{n} e^{-\lambda t_i},$$

the equation (3.1) *has*

$$q_1(t) = -Ae^{\frac{\lambda}{3}t}$$

as a strict upper solution and

$$q_2(t) = Ae^{\frac{(\lambda+1)}{3}t}$$

as a strict lower solutions where $A \neq 0$.

Proof

The differential equation can be re-written as

$$\ddot{q} + \delta\dot{q} - \alpha_0 q + \beta q^3 - N_0 K e^{\lambda t} = 0$$

Now, for $q_1(t) = -Ae^{\frac{\lambda t}{3}}$, the above equation becomes

$$\ddot{q}_1(t) + \delta\dot{q}_1(t) - \alpha_0 q_1(t) + \beta q^3 - N_0 K e^{\lambda t} = -A\frac{\lambda^2}{9}e^{\frac{\lambda t}{3}} - A\frac{\lambda}{3}\delta e^{\frac{\lambda t}{3}} + \alpha_0 A e^{\frac{\lambda t}{3}}$$

$$-\beta A^3 e^{\lambda t} - N_0 K e^{\lambda t}$$

$$= \frac{-A}{9}\left[\lambda^2 + 3\delta\lambda - 9\alpha_0\right]e^{\frac{\lambda t}{3}} - \beta A^3 e^{\lambda t} - N_0 K e^{\lambda t}$$

$$= 0 - N_0 K e^{\lambda t} - N_0 K e^{\lambda t}$$

$$= -2N_o K e^{\lambda t} < 0 \qquad \{\text{ since } N_0 > 0, K > 0 \text{ and } t \geq 0\}$$

Thus, $q_1(t) = -Ae^{\frac{\lambda t}{3}}$ is a strict upper solution of the differential equation (3.1)

Next , consider $q_2(t) = Ae^{\frac{(\lambda+1)}{3}t}$. Substituting $q(t) = q_2(t)$ into the differential equation, we get

$$\ddot{q}_2(t) + \delta\dot{q}(t) - \alpha_0 q_2(t) + \beta q_2^3(t) - N_0 K e^{\lambda t} = \frac{(\lambda+1)^2}{9}Ae^{\frac{(\lambda+1)t}{3}} +$$

$$\delta A\frac{(\lambda+1)}{3}e^{\frac{(\lambda+1)t}{3}} - \alpha_0 A e^{\frac{(\lambda+1)t}{3}} + \beta A^3 e^{(\lambda+1)t} - N_0 K e^{\lambda t}$$

$$= \frac{A}{9}\left(\lambda^2 + 2\lambda + 1\right)e^{\frac{(\lambda+1)t}{3}} + \frac{A\lambda}{3}\delta e^{\frac{(\lambda+1)t}{3}} + \frac{A\delta}{3}e^{\frac{(\lambda+1)t}{3}} -$$

$$\alpha_0 A e^{\frac{(\lambda+1)t}{3}} + \beta A^3 e^{(\lambda+1)t} - N_0 K e^{\lambda t}$$

$$= \frac{A}{9}\left(\lambda^2 + 3\lambda\delta - 9\alpha_0\right)e^{\frac{(\lambda+1)t}{3}} + \frac{A}{9}(2\lambda + 1 + 3\delta)e^{\frac{(\lambda+1)t}{3}} + N_0 K e^{(\lambda+1)t} - N_0 K e^{\lambda t}$$

$$= 0 + \frac{A}{9}(2\lambda + 3\delta + 1)e^{\frac{(\lambda+1)t}{3}} + N_0 K e^{\lambda t}\left(e^t - 1\right)$$

$$> 0 \qquad \{\text{since } N_0, K, A > 0 \text{ and } t \geq 0\}.$$

$$\Rightarrow \ddot{q}_2(t)+\delta\dot{q}_2(t)-\alpha_0 q_2(t)+\beta q_2^3(t)-N_0 K e^{\lambda t}>0$$

Hence, $q_2(t)=Ae^{\frac{(\lambda+1)t}{3}}$ is a strict lower solution of the differential equation (3.1). And, the inequality $-Ae^{\frac{\lambda t}{3}}<Ae^{\frac{\lambda t}{3}}\le Ae^{\frac{(\lambda+1)t}{3}}$ shows that $q_1(t)\le q_2(t)$

Having shown that our model (3.1) satisfies all the conditions in theorem 3.1, then we conclude that our model (3.1) has at least one unstable T-periodic solution $\hat{S}$, with $q_1\le\hat{S}\le q_2$ provided that the number of T-periodic solutions is finite. Our next task is to interpret this result and see how we can apply it to predict earthquake occurrence.

Theorem 3.1 can be interpreted in terms of seismic time series in this way. If the solution lies between the primary (maximum) peak and the lower (minimum) peak, there is likely going to be at least one earthquake occurrence and we can adduce that equation (3.1) has at least one asymptotically unstable *T*- periodic solution $\hat{q}$.

Note

1. The assumption

$$\lambda^2+3\delta\lambda-9\alpha_0=0 \text{ and } \beta A^3=N_0K, \quad \text{where } K=\sum_{i=1}^{n}e^{-\lambda t_i}$$

is the implication of the solution *q*(*t*) = $Ae^{\frac{\lambda t}{3}}$ where $A\ne 0$ of our model

$$\ddot{q}+\delta\dot{q}-\alpha_0 q+\beta q^3=N_0 e^{\lambda t}\sum_{i=1}^{n}e^{-\lambda t_i},$$

where $\lambda,\beta,\alpha_0,N_0>0$.

If we substitute $q(t)=Ae^{\frac{\lambda t}{3}}$ into our model, we get

$$\frac{A}{9}\left(\lambda^2+3\delta\lambda-9\alpha_0\right)e^{\frac{\lambda t}{3}}+\beta A^3 e^{\lambda t}-N_0 K e^{\lambda t}=0$$

provided $\lambda^2+3\delta\lambda-9\alpha_0=0 \quad and \quad \beta A^3=N_0K$, where $K=\sum_{i=1}^{n}e^{-\lambda t_i}$.

The first assumption $\lambda^2+3\lambda\delta-9\alpha_0=0$ is valid , since it is a quadratic equation with two distinct real roots λ_1,λ_2 of which one of the roots satisfy the condition $\lambda>0$ in our model. We can solve the quadratic equation to get

$$\lambda_1=\frac{-3\delta+3\sqrt{\delta^2+4\alpha_0}}{2} \quad =3r_1$$

$$\lambda_2=\frac{-3\delta-3\sqrt{\delta^2+4\alpha_0}}{2} \quad =3r_2$$

But we know that

$$-\delta+\sqrt{\delta^2+4\alpha_0} \quad >0\Rightarrow\frac{-3\delta+3\sqrt{\delta^2+4\alpha_0}}{2} \quad >0 \quad .$$

Thus, we have that $\lambda_1>0$. Therefore we conclude that the assumption

$$\lambda^2 + 3\delta\lambda - 9\alpha_0 = 0$$

is compatible with the condition $\lambda > 0$ in our model.

2. Also the second assumption

$$\beta A^3 = N_0 K,\ A \neq 0 \quad \text{where} \quad K = \sum_{i=1}^{n} e^{-\lambda t}$$

as we have seen helped us to simplify our problem, since by this assumption we can add $\beta A^3 e^{\lambda t} = N_0 K e^{\lambda t}$ together. This second assumption does not change the + (positive) nature of our model neither does it affect our result.

4.0 Conclusion

Microseisms have been widely studied, both observationally and theoretically, through its power spectra, with the aim of improving the detestability of the arrival time of seismic waves that signal the occurrence of earthquake.

In this present study we have focused our interest on the microseism time series model with coda waves as the main source of energy and have shown that it is a good model for predicting earthquake occurrence.

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