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On the equivalence of some iteration schemes for a class of quasi-contractive operators

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Abstract

***We prove that Picard, Mann, Mann with errors, Ishikawa, Ishikawa with errors, Noor, Noor with errors, multistep and multistep iterations with errors are all equivalent when applied to a class of quasi-contractive operators. Our results are extensions and generalisations of the known results of Soltuz* [11, 12].**

Keywords: Picard, Mann, Ishikawa, Noor, Multistep, iterations with their respective errors, quasi-contractive operators.

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1.0 Introduction

Let X be a real normed space, K a non-empty, convex subset of X and T a self map in K. Let $x_0 \in K$. The *Picard iteration* (see [1]) is defined by

$$x_{n+1} = Tx_n,\ n \geq 0 \tag{1.1}$$

For any given u_0, $x_0 \in K$, the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$u_{n+1} = a_n^1 u_n + b_n^1 T u_n + c_n^1 s_n^1 \tag{1.2}$$

where $\{a_n^1\}$, $\{b_n^1\}$ and $\{c_n^1\}$ are sequences in [0, 1] such that

$$a_n^1 + b_n^1 + c_n^1 = 1 \text{ for all } n \geq 0,\ \{s_n^1\} \text{ is a bounded sequence in K and } \sum_{n=0}^{\infty} b_n^1 = \infty$$

is called the *Mann iterative scheme with errors* (see [13]).
If $c_n = 0$ for every n in (1.2), we have the *Mann iterative scheme* (without errors) as

$$x_{n+1} = a_n^1 x_n + b_n^1 T x_n \tag{1.3}$$

And if $c_n \equiv 1$ in (1.2), we have the Mann iterative scheme with errors in the sense of Liu [6] as:

$$x_{n+1} = a_n^1 x_n + b_n^1 T x_n + s_n^1 \tag{1.3b}$$

The sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} u_{n+1} &= a_n^2 u_n + b_n^2 T u_n^1 + c_n^2 s_n^2 \\ u_n^1 &= a_n^1 u_n + b_n^1 T u_n + c_n^1 s_n^1,\ n \geq 0 \end{aligned} \tag{1.4}$$

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where $\{s_n^2\}_{n=0}^{\infty}$, $\{s_n^1\}_{n=0}^{\infty}$ are bounded sequences in K and $\{a_n^2\}_{n=0}^{\infty}$, $\{b_n^2\}_{n=0}^{\infty}$, $\{c_n^2\}_{n=0}^{\infty}$, $\{a_n^1\}_{n=0}^{\infty}$, $\{b_n^1\}_{n=0}^{\infty}$, $\{c_n^1\}_{n=0}^{\infty}$ are real sequences in [0,1] such that $a_n^1 + b_n^1 + c_n^1 = a_n^2 + b_n^2 + c_n^2 = 1$ for all $n \geq 0$ and $\sum_{n=0}^{\infty} b_n^1 = \infty$, is called *Ishikawa iterative scheme with errors* (see [13]).

If $c_n^2 = c_n^1 = 0$ in (1.4), we have the *Ishikawa iterative scheme* (without errors) as

$$\begin{aligned} x_{n+1} &= a_n^2 x_n + b_n^2 T x_n^1 \\ x_n^1 &= a_n^1 x_n + b_n^1 T x_n, \; n \geq 0 \end{aligned} \tag{1.5}$$

If $c_n^2 = c_n^1 = 1$ in (1.4), we have the Ishikawa iteration with errors in the sense of Liu [4] as

$$\begin{aligned} x_{n+1} &= a_n^2 x_n + b_n^2 T x_n^1 + s_n^2 \\ x_n^1 &= a_n^1 x_n + b_n^1 T x_n + s_n^1, \; n \geq 0 \end{aligned} \tag{1.5b}$$

In [9], the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} u_{n+1} &= a_n^3 u_n + b_n^3 T u_n^2 + c_n^3 s_n^3 \\ u_n^2 &= a_n^2 u_n + b_n^2 T u_n^1 + c_n^2 s_n^2 \\ u_n^1 &= a_n^1 u_n + b_n^1 T u_n + c_n^1 s_n^1, \; n \geq 0 \end{aligned} \tag{1.6}$$

where $\{s_n^1\}_{n=0}^{\infty}$, $\{s_n^2\}_{n=0}^{\infty}$ and $\{s_n^3\}_{n=0}^{\infty}$ are bounded sequences in K and $\{a_n^i\}_{n=0}^{\infty}$, $\{b_n^i\}_{n=0}^{\infty}$, $\{c_n^i\}_{n=0}^{\infty}$, are bounded sequences in [0, 1] such that $a_n^i + b_n^i + c_n^i = 1$ for each $i \in [1,2,3]$, n ≥ 0 and $\sum_{n=0}^{\infty} b_n^1 = \infty$ is called the *Noor iterative scheme with errors*.

If $c_n = c_n^1 = c_n^2 = 0$ in (1.6), we have the *Noor iterative scheme* (without errors) as

$$\begin{aligned} x_{n+1} &= a_n^3 x_n + b_n^3 T x_n^2 \\ x_n^2 &= a_n^2 x_n + b_n^2 T x_n^1 \\ x_n^1 &= a_n^1 x_n + b_n^1 T x_n \end{aligned} \tag{1.7}$$

In [14], the *multistep iterative scheme with errors* is defined as

$$\begin{aligned} u_{n+1} &= a_n^N u_n + b_n^N T u_n^{N-1} + c_n^N s_n^N \\ u_n^{N-1} &= a_n^{N-1} u_n + b_n^{N-1} T u_n^{N-2} + c_n^{N-1} s_n^{N-1} \\ &\vdots \qquad \vdots \qquad \vdots \\ u_n^3 &= a_n^3 u_n + b_n^3 T u_n^2 + c_n^3 s_n^3 \\ u_n^2 &= a_n^2 u_n + b_n^2 T u_n^1 + c_n^2 s_n^2 \\ u_n^1 &= a_n^1 u_n + b_n^1 T u_n + c_n^1 s_n^1 \end{aligned} \tag{1.8}$$

where $\{s_n^1\}_{n=0}^{\infty}$, $\{s_n^2\}_{n=0}^{\infty}$, …, $\{s_n^N\}_{n=0}^{\infty}$ are bounded sequences in K, $\{a_n^i\}_{n=0}^{\infty}$, $\{b_n^i\}_{n=0}^{\infty}$, …, $\{c_n^i\}_{n=0}^{\infty}$ are sequences in [0, 1] such that $a_n^i + b_n^i + c_n^i = 1$ for $i \in [1,2,...,N]$ and $\sum_{n=0}^{\infty} b_n^1 = \infty$.

If we set $c_n^1 = c_n^2 = ... = c_n^N = 0$ in (1.8), then it reduces to the *multistep iterative scheme (without errors).*

$$\begin{aligned} x_{n+1} &= a_n^N x_n + b_n^N Tx_n^{N-1} \\ x_n^{N-1} &= a_n^{N-1} x_n + b_n^{N-1} Tx_n^{N-2} \\ &\vdots \\ x_n^3 &= a_n^3 x_n + b_n^3 Tx_n^2 \\ x_n^2 &= a_n^2 x_n + b_n^2 Tx_n^1 \\ x_n^1 &= a_n^1 x_n + b_n^1 Tx_n \end{aligned} \tag{1.9}$$

Remark **1.1**

(i) If we set $N = 3$, in (1.8), we have the Noor Iteration scheme with erros (1.6)

(ii) If we set N = 3, with $c_n^1 = c_n^2 = c_n^3 = 0$ in (1.8), we have the Noor iteration scheme (1.7).

(iii) If we set N = 2 in (1.8), we have the Ishikawa iteration with errors (1.4).

(iv) If we set N = 2, with $c_n^1 = c_n^2 = 0$ in (1.8), we have the Ishikawa iteration (1.5)

(v) If we set N = 1 in (1.8), we have the Mann iteration with errors (1.2).

(vi) If we set N = 1, with $c_n^1 = 0$ in (1.8), we have the Mann iteration (1.3).

Definition **1.1 [15]**

The operator $T : X \to X$ is a Zamfirescu operator if and only if there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b$, $c < \frac{1}{2}$, such that for each pair x, $y \in X$, at least one of the conditions is true.

(Z_1) $\| Tx - Ty \| \le a \| x - y \|$

(Z_2) $\| Tx - Ty \| \le b(\| x - Tx \| + \| y - Ty \|)$

(Z_3) $\| Tx - Ty \| \le c(\| x - Ty \| + \| y - Tx \|)$

It is easy to show that every Zamfirescu operator T satisfies the inequality

$$\| Tx - Ty \| \le \delta \| x - y \| + 2\delta \| x - Tx \| \tag{1.10}$$

where $\delta = \max\left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\} < 1$.

Definition **1.2 [2]**

Let (X, d) be a metric space. A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx, Ty) \le ad(x, y), \text{ for all } x,\ y \in X, \text{ where } 0 < a < 1 \tag{1.11}$$

The map T is called a Kannan mapping [5] if there exists $b \in \left(0, \frac{1}{2}\right)$, such that

$$d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)], \text{ for all } x,\ y \in X \tag{1.12}$$

The map T is called a Chatterjea mapping [3] if there exists $c \in \left(0, \frac{1}{2}\right)$, such that

$$d(Tx,Ty) \le c[d(x,Tx) + d(y,Ty)], \text{ for all } x,\ y \in X \tag{1.13}$$

Combining definitions (1.11), (1.12) and (1.13), we have (1.10).

***Definition* 1.3 [1]**

Let (X,d) be a metric space. A mapping $T : X \to X$ is called a quasi-contraction if

$$d(Tx,Ty) \le h \max[d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)], \tag{1.14}$$

where $0 < h < 1$.

***Remark* 1.2.**

Definition 1.1 is a subclass of Definition 1.3. Olaleru [10] considered a class of quasi-contractive operators introduced by Osilike (see [1]) and defined thus:

***Definition* 1.4 [10]:** Let (X,d) be a metric space, there exist $L \ge 0$, $a \in [0,1)$ such that for each x, $y \in X$,

$$d(Tx,Ty) \le Ld(x,Tx) + ad(x,y) \tag{1.15}$$

Observe that quasi-contraction maps (1.14) are independent of the class of quasi-contractive maps (1.15), while the class of quasi-contractive maps satisfying $(Z_1 - Z_3)$ are also quasi-contractive operators defined in (1.15).

In a recent paper, Soltuz [11] shows that the Picard, Mann and Ishikawa iterations are equivalent when dealing with quasi-contractive operators. In [12], it was also shown that for the same class of operators, Krasnoselskij, Mann, Ishikawa, Noor and multistep iterations are equivalent.

In [8, page 145], Rafiq states that the iterative schemes with errors introduced by Liu [6] are not satisfactory. The errors can occur in a random way. The conditions imposed on the error terms which say that they tend to zero as n tends to infinity are therefore, unreasonable. Better results are obtained if we use the iterative schemes introduced by Xu [13].

In 2007, Zhiqun [16] makes some remark on the equivalence of Picard, Mann and Ishikawa iterations for the class of quasi-contractive operators.

It is the main purpose of this paper to show that Picard, Mann, Mann with errors, Ishikawa, Ishikawa with errors, Noor, Noor with errors, multistep and multistep with errors are all equivalent for a class of quasi-contractive operators. Our results extend and generalise the results of Soltuz [11, 12].

***Lemma* 1.5 [8]**

Let $\{\theta\}_{n \ge 0}$ be a non-negative sequence which satisfies the following inequality

$\theta_{n+1} \le (1-\lambda_n)\theta_n + \delta_n + \gamma_n$ where $\lambda_n \in (0, 1)$, $\gamma_n \ge 0$, for every $n \in N$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\sum_{n=0}^{\infty} \gamma_n = \infty$ and $\gamma_n = o(\lambda_n)$, then $\lim_{n\to\infty} \theta_n = 0$.

2.0 Main Results

Theorem 2.1

Let X be a normed space, K a non-empty, convex, closed subset of X and $T : K \to K$ an operator satisfying the Zamfirescu conditions $(Z_1 - Z_3)$. Let $u_0 = x_0 \in K$ and define $\{u_n\}$ and $\{x_n\}$ as sequences satisfying respectively (1.8) *and* (1.9) *with the conditions therein. If $\lim_{n\to\infty} c_n^i = 0$ for each i, then the following are equivalent:*

(i) *The multistep iteration* (1.9) *converges to* p.

(ii) *The multistep iteration with errors* (1.8) *converges to* p.

Proof

We prove that (i) $\Rightarrow$ (ii) using (1.8), (1.9) and (1.10) with $x = x_n$, $y = u_n$. $\| x_n^1 - u_n^1 \| =$

$$\| a_n^1 x_n + b_n^1 Tx_n - (a_n^1 u_n + b_n^1 Tu_n + c_n^1 s_n^1) \|$$

$$\le a_n^1 \| x_n - u_n \| + b_n^1 \| Tx_n - Tu_n \| + c_n^1 \| s_n^1 \|$$

$$\le a_n^1 \| x_n - u_n \| + b_n^1 \delta \| x_n - u_n \| + 2b_n^1 \delta \| x_n - Tx_n \| + c_n^1 \| s_n^1 \|$$

$$= (a_n^1 + b_n^1 \delta) \| x_n - u_n \| + 2b_n^1 \delta \| x_n - Tx_n \| + c_n^1 \| s_n^1 \| \quad (1.16)$$

$$\le [1 - b_n^1 (1-\delta)] \| x_n - u_n \| + 2b_n^1 \delta \| x_n - Tx_n \| + c_n^1 \| s_n^1 \| \quad (1.17)$$

$$= (1 - w_n^1)\theta_n + r_n^1 + t_n^1$$

where $w_n^1 = b_n^1 (1-\delta)$, $r_n^1 = m_n^1 v_n$, $m_n^1 = 2b_n^1 \delta$, $v_n = \| x_n - Tx_n \|$, $t_n^1 = c_n^1 \| s_n^1 \|$.

Using (1.17) and (1.10), we have

$$\| x_n^2 - u_n^2 \| = \| a_n^2 x_n + b_n^2 Tx_n^1 - (a_n^2 u_n + b_n^2 Tu_n^1 + c_n^2 s_n^2) \|$$

$$\le a_n^2 \| x_n - u_n \| + b_n^2 \| Tx_n - Tu_n \| + c_n^2 \| s_n^1 \|$$

$$\le a_n^2 \| x_n - u_n \| + b_n^2 \delta \| x_n^1 - u_n^1 \| + 2b_n^2 \delta \| x_n^1 - Tx_n^1 \| + c_n^2 \| s_n^2 \|$$

$$\le a_n^2 \| x_n - u_n \| + b_n^2 \delta (1 - b_n^1 (1-\delta)) \| x_n - u_n \|$$

$$+ 2b_n^1 b_n^2 \delta^2 \| x_n - Tx_n \| + 2b_n^2 \delta \| x_n^1 - Tx_n^1 \| + c_n^1 b_n^2 \delta \| s_n^1 \| + c_n^2 \| s_n^2 \|$$

$$= [1 - a_n^2 + b_n^2 \delta - b_n^1 b_n^2 \delta (1-\delta)] \| x_n - u_n \|$$

$$+ 2b_n^1 b_n^2 \delta^2 \| x_n - Tx_n \| + 2b_n^2 \delta \| x_n^1 - Tx_n^1 \| + c_n^1 b_n^2 \delta \| s_n^1 \| + c_n^2 \| s_n^2 \|$$

$$= [1 - b_n^2 (1-\delta) - c_n^2 - b_n^1 b_n^2 \delta (1-\delta)] \| x_n - u_n \|$$

$$+ 2b_n^1 b_n^2 \delta^2 \| x_n - Tx_n \| + 2b_n^2 \delta \| x_n^1 - Tx_n^1 \| + c_n^1 b_n^2 \delta \| s_n^1 \| + c_n^2 \| s_n^2 \|$$

$$\le [1 - b_n^2 (1-\delta)] \| x_n - u_n \| + 2b_n^1 b_n^2 \delta^2 \| x_n - Tx_n \|$$

$$+ 2b_n^2 \delta \| x_n^1 - Tx_n^1 \| + c_n^1 b_n^2 \delta \| s_n^1 \| + c_n^2 \| s_n^2 \| \quad (1.18)$$

$$= (1 - w_n^2)\theta_n + r_n^2 + t_n^2$$

where $w_n^2 = b_n^2 (1-\delta)$, $r_n^2 = m_n^2 v_n^1$, $m_n^2 = 2b_n^2 \delta$, $v_n^1 = b_n^1 \delta \| x_n - Tx_n \| + \| x_n^1 - Tx_n^1 \|$, $t_n^2 = c_n^1 b_n^2 \delta \| s_n^1 \| + c_n^2 \| s_n^2 \|$.

Using (1.18) and (1.10), we have $\| x_n^3 - u_n^3 \| = \| a_n^3 x_n + b_n^3 Tx_n^2 - (a_n^3 u_n + b_n^3 Tu_n^2 + c_n^3 s_n^3) \|$

$$\le a_n^3 \| x_n - u_n \| + b_n^3 \| Tx_n^2 - Tu_n^2 \| + c_n^3 \| s_n^3 \|$$

$$\le a_n^3 \| x_n - u_n \| + b_n^3 \delta \| x_n^2 - u_n^2 \| + 2b_n^3 \delta \| x_n^2 - Tx_n^2 \| + c_n^3 \| s_n^3 \|$$

$$\le a_n^3 \| x_n - u_n \| + b_n^3 \delta (1 - b_n^2 (1-\delta)) \| x_n - u_n \|$$

$$+ 2b_n^1 b_n^2 b_n^3 \delta^3 \| x_n - Tx_n \| + 2b_n^2 b_n^3 \delta^2 \| x_n^1 - Tx_n^1 \|$$

$$+ 2b_n^3 \delta \| x_n^2 - Tx_n^2 \| + c_n^1 b_n^2 b_n^3 \delta^2 \| s_n^1 \| + c_n^2 b_n^3 \delta \| s_n^2 \| + c_n^3 \| s_n^3 \|$$

$= [1-b_n^3-c_n^3+b_n^3\delta(1-b_n^2(1-\delta))] \| x_n-u_n \|$
$+2b_n^1b_n^2b_n^3\delta^3 \| x_n-Tx_n \| +2b_n^2b_n^3\delta^2 \| x_n^1-Tx_n^1 \|$
$+2b_n^3\delta \| x_n^2-Tx_n^2 \| +c_n^1b_n^2b_n^3\delta^2 \| s_n^1 \| +c_n^2b_n^3\delta \| s_n^2 \| +c_n^3 \| s_n^3 \|$
$\leq [1-b_n^3(1-\delta)] \| x_n-u_n \| +2b_n^1b_n^2b_n^3\delta^3 \| x_n-Tx_n \|$
$+2b_n^2b_n^3\delta^2 \| x_n^1-Tx_n^1 \| +2b_n^3\delta \| x_n^2-Tx_n^2 \|$
$+c_n^1b_n^2b_n^3\delta^2 \| s_n^1 \| +c_n^2b_n^3\delta \| s_n^2 \| +c_n^3 \| s_n^3 \| = [1-b_n^3(1-\delta)] \| x_n-u_n \|$
$+2b_n^3\delta[b_n^1b_n^2\delta^2 \| x_n-Tx_n \| +b_n^2\delta \| x_n^1-Tx_n^1 \| + \| x_n^2-Tx_n^2 \|]$
$+c_n^1b_n^2b_n^3\delta^2 \| s_n^1 \| +c_n^2b_n^3 \| s_n^2 \| +c_n^3 \| s_n^3 \| = (1-w_n^3)\theta_n+r_n^3+t_n^3$

where $w_n^3=b_n^3(1-\delta)$, $r_n^3=m_n^3v_n^2$, $m_n^3=2b_n^3\delta$,
$v_n^2=b_n^1b_n^2\delta^2 \| x_n-Tx_n \| +b_n^2\delta \| x_n^1-Tx_n^1 \| + \| x_n^2-Tx_n^2 \|$,
$t_n^3=c_n^1b_n^2b_n^3\delta^2 \| s_n^1 \| +c_n^2b_n^3\delta \| s_n^2 \| +c_n^3 \| s_n^3 \|$.

Continuing the above process, we have
$\| x_{n+1}-u_{n+1} \| \leq [1-b_n^N(1-\delta)] \| x_n-u_n \| +r_n^N+t_n^N = (1-w_n^N)\theta_n+r_n^N+t_n^N$
where $w_n^N=b_n^N(1-\delta)$, $\theta_n = \| x_n-u_n \|$, $r_n^N=m_n^Nv_n^{N-1}$, $m_n^N=2b_n^N\delta$,
$v_n^{N-1}=b_n^{N-2}b_n^{N-1}\delta^{N-1} \| x_n-Tx_n \| +b_n^{N-1}\delta^{N-2} \| x_n^{N-2}-Tx_n^{N-2} \| + \| x_n^{N-1}-Tx_n^{N-1} \|$,
$t_n^N=c_n^{N-2}b_n^{N-1}b_n^N\delta^{N-1} \| s_n^{N-2} \| +c_n^{N-1}b_n^N\delta^{N-2} \| s_n^{N-1} \| +c_n^N \| s_n^N \|$.
By Lemma 1.5, let
$w_n^N=\lambda_n$, $\theta_n = \| x_n-u_n \|$, $\delta_n=r_n^N$, $\gamma_n=t_n^N$, $\Rightarrow \qquad \theta_{n+1} \leq (1-\lambda_n)\theta_n+\delta_n+\gamma_n$
$\gamma_n \geq 0$, $\sum_{n=0}^{\infty}\lambda_n=\infty$, $\sum_{n=0}^{\infty}\gamma_n=\infty$ and $\delta_n=o(\lambda_n)$
Hence, the assumptions in Lemma 1.5 are satisfied, $\Rightarrow \lim_{n\to\infty}\theta_n=0$. Next, we show that $\lim_{n\to\infty}u_n=p$.
By assumption, $\lim_{n\to\infty}x_n=p$, $p\in F(T)$. $\| u_n-p \| \leq \| u_n-x_n \| + \| x_n-p \|$
$\Rightarrow \qquad \lim_{n\to\infty} \| u_n-p \| = 0$
Hence, $\lim_{n\to\infty}u_n=p$.
This shows that (i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (i) is obvious if we let $c_n^1=c_n^2=\ldots=c_n^N=0$ in (1.8).

Therefore, the convergence of multistep iteration and multistep iteration with errors are equivalent for a class of quasi-contractive operators.
Theorem 2.1 leads to the following corollaries.
Corollary *2.1*

Let X be a normed space, K a non-empty, convex, closed subset of X and $T:K\to K$ an operator satisfying the Zamfirescu conditions (Z_1-Z_3). Let $u_0=x_0\in K$ and define $\{u_n\}$ respectively by

(1.2), (1.4), (1.6), (1.8); and *{* x_n *}* respectively by *(1.3), (1.5), (1.7)* and (*1.9*). *If* $\lim_{n\to\infty} c_n^i$ *= 0* for each i, then the following are equivalent:

(A) (i) *The Noor iteration* (1.7) *converges to* p *.*

(ii) *The Noor iteration with errors* (1.6) *converges to* p *.*

(B) (i) *The Ishikawa iteration* (1.5) *converges to* p *.*

(ii) *The Ishikawa iteration with errors* (1.4) *converges to* p *.*

(C) (i) *The Mann iteration* (1.3) *converges to* p *.*

(ii) *The Mann iteration with errors* (1.2) *converges to* p *.*

Theorem 2.2 ***[15, Thm 2.1]***

*Le*t E be a normed space, D a non-empty, convex, closed subset of E and $T: D \to D$ a Zamfirescu operator. Suppose that T has a fixed point $q \in D$. Let $\{p_n\}_{n=0}^{\infty}$ be defined by *(1.1)* for $p_0 \in K$ and let $\{u_n\}_{n=0}^{\infty}$ be defined by (1.2) for $u_0 \in K$, then the following are equivalent:

(i) *The Picard iteration* (1.1) *converges to the fixed point of* T *.*

(ii) *The Mann iteration* (1.2) *converges to the fixed point of* T *.*

Theorem ***2.3 [15, Thm 2.3]***

Let E be a normed space, D a non-empty, convex, closed subset of E and $T: D \to D$ a Zamfirescu operator. Suppose that T has a fixed point $q \in D$. Let $\{p_n\}_{n=0}^{\infty}$ be defined by (1.1) for $p_0 \in K$ and let $\{x_n\}_{n=0}^{\infty}$ be defined by *(1.3)* for $x_0 \in K$, then the following are equivalent:

(i) The Picard iteration (1.1) converges to the fixed point of *T*.

(ii) The Ishikawa iteration (1.3) converges to the fixed point of *T*.

Theorem **2.4 [11, Thm 3]**

Let *X* be a normed space, *X* a non-empty, convex, closed subset of *X* and *T: D* → *D* an operator satisfying the conditions ($Z_1 – Z_3$). If $u_0 = x_0$ Є *K*, then the following are equivalent:

(i) *The Mann iteration* (1.3) *converges to the fixed point of T.*

(ii) *The multistep iteration* (1.9) *converges to the fixed point of T.*

Corollary **2.2 [11, Cor 2]**

Let *X* be a normed space, *D* a non-empty, convex, closed subset of *X* and *T: D* → *D* an operator satisfying the condition *Z*. If the initial point is the same for all iterations, $\alpha_n \geq A > 0 \;\; \forall \;\; n \in N$, then the following are equivalent*:*

(i) *The Mann iteration* (1.3) *converges to the fixed point of T.*

(ii) *The Ishikawa iteration* (1.5) *converges to the fixed point of T.*

(iii) *The Noor iteration* (1.7) *converges to the fixed point of T.*

(iv) *The multistep iteration* (1.9) *converges to the fixed point of T.*

In view of Theorem 2.1, Corollaries 2.1 and 2.2, Theorems 2.2 and 2.3, we have the following:

Corollary **2.3**

Let X be a normed space, K *a non-empty, convex, closed subset of X and T: K →K an operator satisfying the Zamfirescu conditions* ($Z_1 – Z_3$)*.. Let* $u_0 = x_0$ *Є K, and define {* u_n *} respectively by (1*.2), (1.4), (1.6), (1.8) and *{* x_n *}* respectively by (1.1), (1.3), (1.5), (1.7) and (1.9). If $\lim_{n\to\infty} c_n^i = 0$ for all *i*, then the following are equivalent:

(i) *The Picard iteration* (1.1) *converges to the fixed point of T.*

(ii) *The Mann iteration* (1.3) *converges to the fixed point of T.*

(iii) *The Mann iteration with errors* (1.2) *converges to the fixed point of T.*

(iv) *The Ishikawa iteration* (1.5) *converges to the fixed point of T.*

(v) *The Ishikawa iteration with errors* (1.4) *converges to the fixed point of T.*
(vi) *The Noor iteration* (1.7) *converges to the fixed point of T.*
(vii) *The Noor iteration with errors* (1.6) *converges to the fixed point of T.*
(viii) *The multistep iteration* (1.9) *converges to the fixed point of T.*
(ix) *The multistep iteration with errors* (1.8) *converges to the fixed point of T.*

3.0 Conclusion

Our result extends and generalizes the result of Soltuz [11,12] in the sense that it considered the various iterations schemes with their errors, whereas the author [11,12] considered them without their errors. Olaleru [10] did not consider the Noor and Multistep iterations for this class of operators

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