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Egyptian fractions and practical numbers

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Abstract

It is easily seen that if p can be written as the sum of distinct divisors of q, then the fraction $\frac{p}{q}$ can be expanded with no denominator greater than q itself. The idea behind this concept has provided another method for writing the Egyptian fractions (sum of Unit fractions) for any rational $\frac{p}{q}$, $q \neq 0\cdot$

1.0 Introduction

***Definition* 1.1**

A practical number is an integer N such that for all values of n < N, n can be written as the sum of distinct divisors of N i.e. N is practical if every n with 1 ≤ n ≤ N is a sum of distinct divisors of N.

For example, the numbers 4 and 12 are practical numbers as shown below:

N = 4(n = 1, 2, 3)	N = 12(n = 1, 2, …, 11)
1 = 1	1 = 1
2 = 2	2 = 2
3 = 2 + 1	3 = 2 + 1
	4 = 3 + 1
	5 = 4 + 1
	6 = 4 + 2
	7 = 4 + 3
	8 = 6 + 2
	9 = 6 + 3
	10 = 6 + 4
	11 = 6 + 2 + 3

Table 1.1

On the other hand, 10 is not a practical number because

$$N = 10(n = 1,\ 2, \ldots, 9)$$

1 = 1
2 = 2
4 = 3 + 1 or 4 = 2 + 2

for 4 = 3 + 1, 3 is not a divisor of 10 and for 4 = 2 + 2, the two numbers are not distinct. Below is the list of the first twenty practical numbers

1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 54, 56, 60

***Definition* 1.2**

An Egyptian Fraction is a sum of positive (Usually) distinct unit fractions e.g. $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$ *(no unit fraction can be repeated* [1]).

This work attempt to marry this two definitions with the purpose of being able to decompose the fraction $\frac{p}{q}$ into its Egyptian Fraction form. For example, if we want to expand $\frac{9}{20}\cdot$ Note that 9 can be written as a distinct divisors of 20 i.e. 9 = 4 + 5, so

$$\frac{9}{20} = \frac{4+5}{20} = \frac{1}{5} + \frac{1}{4}$$

Also

$$\frac{19}{28} = \frac{14+4+1}{28} = \frac{1}{2} + \frac{1}{7} + \frac{1}{28}$$

Next we consider a theorem that connects practical numbers and Egyptian fractions.

***Theorem* 1.1**

$\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x^2} + \Lambda + \frac{1}{x_k}$ *if and only if there exist positive integers* M *and* N *and divisors* $D_1,\ D_2, \Lambda\ , D_k$ *of* N *such that* $\frac{M}{N} = \frac{m}{n}$ *and* $D_1 + D_2 + \Lambda + D_k = 0$ *(mod M). Also, the last condition can be replaced by* $D_1 + D_2 + \Lambda + D_k = M$ *and the condition* $(D_1 + D_2 + \Lambda + D_k) = 1$ *may be added without affecting the validity of the theorem.*

Proof

First, suppose *M* and *N* exist which satisfy given conditions. Then we simply have

$$\frac{m}{n} = \frac{M}{N} = \frac{D_1 + D_2 + \Lambda + D_k}{cN} = \frac{1}{\frac{cN}{D_1}} + \frac{1}{\frac{cN}{D_2}} + \Lambda + \frac{1}{\frac{cN}{D_k}} \tag{1.1}$$

On the other hand, suppose $\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \Lambda + \frac{1}{x_k}$ is solvable. Then

$$\frac{m}{n} = \sum_{i=1}^{k} \frac{1}{x_i} = \frac{\sum_{i=1}^{k} x_1 \Lambda\ x_{i-1} x_{i+1} \Lambda\ x_k}{x_1, x_2 \Lambda\ x_k} = \frac{M}{N} \tag{1.2}$$

Clearly, then $M = D_1 + D_2 + \Lambda + D_k$, where the D_i all divide $N\cdot$ And we are done. If $(D_1,\ D_2, \Lambda\ , D_k) = d \neq 1,$ then we simply take $\frac{M}{d}$ *and* $\frac{N}{d}$ instead.

Also relating to Egyptian fractions, is an important property of practical numbers which was proved in [2]. Here we, state the theorem without proof.

***Theorem* 1.2**

If n *is a practical number and* q *is any number relatively prime with* n *and* $q < 2n,$ *then* qn *is also practical.*

Proof

Applying theorem 1.2, we expand $\frac{5}{23}\cdot$ First we note that 12 is practical and thus,

$$\frac{5}{23}=\frac{5(12)}{23(12)}$$

Since 23 < 2(12) and 12 is practical, we know that 23(12) is also practical by theorem 1.2. So 5(12) can be written as the sum of distinct divisors of 23(12).

(i) $$\frac{5(12)}{23(12)}=\frac{60}{276} = \frac{46+12+2}{276} = \frac{46}{276}+\frac{12}{276}+\frac{2}{276} = \frac{1}{6}+\frac{1}{23}+\frac{1}{138}$$

(ii) Expand $\frac{7}{31} \Rightarrow \frac{7}{31} = \frac{7(16)}{31(16)}$

Since $31 < 2(16)$ *and* 6 is practical, then $31(16)$ is practical. So we can write $7(6)$ as the sum of distinct divisors of $31(16)$. Thus

$$\frac{7}{31} = \frac{7(16)}{31(16)} = \frac{62+31+16+2+1}{31(16)} = \frac{62}{31(16)}+\frac{31}{31(16)}+\frac{16}{31(16)}+\frac{2}{31(16)}+\frac{1}{31(16)}$$
$$= \frac{1}{8}+\frac{1}{16}+\frac{1}{31}+\frac{2}{248}+\frac{1}{496}$$

2.0 Properties of practical numbers

Properties of practical numbers relating to Egyptian fractions are summarized below.

(i) If n has divisors $1=d_1<d_2<\Lambda<d_c=n$, then n is practical if and only if

$$\sum_{i=1}^{r} d_r \geq d_{r+1}-1 \text{ for all } r< c-1$$

(ii) If n has a subset of divisors $1=d_1<d_2<\Lambda<d_c=n$ in which each is at most twice the previous divisor; then n is practical.

(iii) If n is practical and m is a natural number $\leq n$ then mn is practical. [3]

(iv) If n is practical and the sum of the divisors of n is at least $n+k$ where k is a non-negative integer, then $n(2n+k+1)$ is practical. [3].

3.0 The practical number algorithm for Egyptian fractions

Step 1: Given $\frac{p}{q}$ in lowest terms

Step 2: set $m=1$

Step 3: If qm is not practical, let $m=m+1$ and repeat step 3; otherwise

Step 4: Write $\frac{p}{q}=\frac{pm}{qm}$ and find the expansion.

Express $\frac{3}{7}$ in Egyptian form. Now $\frac{p}{q}=\frac{3}{7}$

By step 2, let $m = 1$ and by step 3 test if qm is practical so

$\frac{3(1)}{7(1)}$, 7 is not practical, so we let $m = m + 1 = 1 + 1 = 2$.

$\frac{3(2)}{7(2)}$, again 21 is not practical $m = m + 1 = 3 + 1 = 4$

$\frac{3(4)}{7(4)}$

Note that 7 < 24 and 4 is practical, so we write 3(4) as the sum of distinct divisors of 7(4) i.e.

$$\frac{3(4)}{7(4)} = \frac{7+4+1}{7(4)} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$$

Note that in step 3, we can instead Test to see if pm can be written as the sum of distinct divisors of $qm \cdot$ however, in finding asymptotic results, we will have to take the worst case for $p -$ thus, testing for practically is more general.

Clearly, this algorithm will terminate because, if nothing else, we can increment m until we reach $2^k \geq q$ (the binary algorithm) [4].

Also, if we let $M(N) =$ smallest m such that mN is practical then we can say $D(N) \leq N \cdot M(N)$, so if we can find a bound for $M(N)$, we can also find an upper bound for $D(N) \cdot$ The calculation of $M(p)$ in the procedure above is based on the following theorem.

***Theorem* 3.1**

$M(p_i) \leq M(p_j)$ for $i < j$

Proof

Suppose $$M(p_j) = m \qquad (3.1)$$

In the general ease, take a number $n < mp_i < mp_j$. Find r, s such that $n = Sp_i + r$ with $0 \leq r < p_i < p_j$. Since $r < p_j$, we can write r as the sum of distinct divisors of m.

$$S < \frac{(n-r)}{p_i} < \frac{n}{p_i} < m \qquad (3.2)$$

We assume $m < p_j$ (this is clearly true for large enough j). So we can write S as the sum of distinct divisors of m. Thus, since $m \; and \; p_i$ are relatively prime we can write n as the sum of distinct divisors of mp_i therefore, $$M(p_i) \leq m = M\ (p_j) \qquad (3.3)$$

which imply $$M(p_i) \leq M(p_j) \qquad (3.4)$$

4.0 Conclusion

Just as we have the real number system *R*, we introduce the practical number system (*P*) where $P \subseteq R$. It is easily seen that if *P* can be written as the sum of distinct divisors of *q*, then $\frac{p}{q}$ can be expanded with no denominator grater than *q* itself. This and other properties of practical numbers, stated hare makes it easy for fractions such as $\frac{p}{q}$ to be expressed in Egyptian fraction form.

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