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Einstein's equation of motion for a photon in fields exterior to astrophysically real or imaginary spherical mass distributions whose tensor field varies with azimuthal angle only

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Abstract

The metric exterior to an astrophysically real or hypothetical spherical mass distribution whose tensor field varies with azimuthal angle only is used to study the motion of a photon in this gravitational field. The general relativistic equation of motion for a photon moving round this spherical distribution of mass is derived .The second-order differential equation obtained differs from that in Schwarzschild's field.

***Keywords*:** equation of motion, astrophysical, spherical, mass, photon, tensor field, azimuthal angle

1.0 Introduction

In general relativity, light follows a special variety of straightest - possible world –line, so called light–like or null geodesics –a generalization of the straight lines along which light travels in Classical Physics and the invariance of light speed in special relativity [1]. Light passing a massive body is deflected towards the massive body The deflection of light (photon) by the Sun was the third prediction of General Relativity that provided the most famous and dramatic test of the theory. Although the effect itself was so small and had no practical implications, the observation of it seized hold of public imagination and cemented Einstein's reputation as a great physicist [2]. An important example of this is starlight being deflected as it passer the Sun; in consequence, the positions of stars observed in the Sun's vicinity during a solar eclipse appear shifted by up to 1.75 arc seconds. This effect was first measured by a British expedition directed by Sir Arthur Eddington and confirmed with significantly higher accuracy by subsequent measurements [1, 3].

This article uses the metric tensor exterior to spherical distributions of mass whose tensor field varies with azimuthal angle only. Although such gravitational fields may not exist physically, the theoretical analysis presented in this article paves the way for the investigation of complex, physically real spherical distributions of mass. By analyzing hypothetical distributions whose tensor field varies with only one coordinate gives us an idea on how to study astrophysically real fields whose tensor field depends on two or more coordinates of space-time.

2.0 Theoretical analysis

It is well known that using Schwarzschild metric for a space time exterior to a static homogenous spherical body, the General Relativistic equation of motion for a photon [4, 5] is

$$\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2 \tag{2.1}$$

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where u is a radial function defined in terms of r in polar coordinates as

$$u(\phi) = \frac{1}{r(\phi)} \tag{2.2}$$

ϕ is the angular coordinate ,G the universal gravitational constant , M the mass of the static spherical body and c the speed of light in vacuum. Equation (2.1) has been solved by the method of successive approximation to obtain satisfactory results for the total deflection of the photon from its original straight line path.

The metric exterior to astrophysically real or imaginary spherical bodies whose tensor field depends on polar angle only is given in [6] as

$$\begin{aligned}
g_{00} &= \left[1 + \frac{2}{c^2} f(\phi)\right] \\
g_{11} &= -\left[1 + \frac{2}{c^2} f(\phi)\right]^{-1} \\
g_{22} &= -r^2 \\
g_{33} &= -r^2 \sin^2\theta \\
g_{\mu\nu} &= 0; otherwise
\end{aligned}$$

Consequently, the invariant world line element in the exterior region of this spherical body is given by

$$c^2 d\tau^2 = c^2\left[1 + \frac{2}{c^2} f(\phi)\right]dt^2 - \left[1 + \frac{2}{c^2} f(\phi)\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2.3}$$

According to General Relativity, light (photon) moves along a null geodesic [1,4,5]; that is

$$c^2 d\tau^2 = 0 \tag{2.4}$$

Thus, our line element, equation (2.3) for a photon moving in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherical body reduces to;

$$0 = c^2\left[1 + \frac{2}{c^2} f(\phi)\right]dt^2 - \left[1 + \frac{2}{c^2} f(\phi)\right]^{-1} dr^2 - r^2 d\phi^2 \tag{2.5}$$

Now, let ω be a parameter which may be used to follow the motion of the photon in this gravitational field. Dividing equation (2.5) by $d\omega^2$ and denoting differentiation with respect to ω by dot (.) yields;

$$0 = c^2\left[1 + \frac{2}{c^2} f(\phi)\right]\dot{t}^2 - \left[1 + \frac{2}{c^2} f(\phi)\right]^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 \tag{2.6}$$

It is well known that $\dot{t}$ is defined in Schwarzschild field [6] as

$$\dot{t} = a\left[1 + \frac{2}{c^2} f(r)\right]^{-1} \tag{2.7}$$

where a is a constant of motion. Using the time equation of motion for test particles [6], it can be shown that $\dot{t}$ takes the form

$$\dot{t} = a\left[1 + \frac{2}{c^2} f(\phi)\right]^{-1} \tag{2.8}$$

in this gravitational field.

Also, the azimuthal equation of motion for particles of non-zero rest masses in the equatorial plane of a spherical body is given as [4, 5, 6, 7]

$$\dot{\phi} = \frac{b}{r^2} \tag{2.9}$$

where b is another constant of motion. Also, $\dot{r}$ can be written in terms of ϕ as follows

$$\dot{r} = \frac{dr}{d\omega}\text{ , thus } \dot{r} = \frac{dr}{d\phi}\frac{d\phi}{d\omega}$$

and hence

$$\dot{r} = \dot{\phi}\frac{dr}{d\phi} \tag{2.10}$$

Thus substituting equations (2.8), (2.9) and (2.10) into equation (2.6) and rearranging gives

$$\frac{b^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 = a^2c^2 - \left[1 + \frac{2}{c^2}f(\phi)\right]\frac{b^2}{r^2} \tag{2.11}$$

Now, let u be a radial function defined in terms of r in polar coordinates by equation (2.2) then

$$\frac{dr}{d\phi} = -u^{-2}\frac{du}{d\phi} \tag{2.12}$$

Substituting equation (2.12) into equation (2.11) and simplifying gives

$$\left(\frac{du}{d\phi}\right)^2 = \frac{c^2a^2}{b^2} - \left[1 + \frac{2}{c^2}f(\phi)\right]u^2 \tag{2.13}$$

Differentiating both sides of equation (2.13) with respect to ϕ yields

$$\frac{d^2u}{d\phi^2} + \frac{1}{c^2}\frac{df}{d\phi}\left(\frac{du}{d\phi}\right)^{-1}u^2 + \left[1 + \frac{2}{c^2}f(\phi)\right]u = 0 \tag{2.14}$$

Equation (2.14) is the general relativistic equation of motion for a photon moving round a spherical distribution of mass whose tensor field varies with polar angle only. It differs from equation (2.1) obtained from Schwarzschild's static spherical field [2].

3.0 Conclusion

The door is thus open for the derivation of an expression for the total deflection of a photon from its original straight line motion, moving this spherical distribution of mass. Our metric tensor in this gravitational field can equally be used to study the motion of particles of non-zero rest masses. The covariant metric tensor can equally be used to derive Einstein's field equations in this astrophyically real or hypothetical distributions of mass.

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