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Batch arrival discrete time queue with server vacation

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Abstract

A class of single server vacation queues, which have batch arrivals and single server, is considered in discrete time. Here the server goes on vacation of random length as soon as the system becomes empty. On return from vacation, if he finds any customers waiting in the queue, the server starts serving the customers one by one until the queue size is zero (the queue discipline is FIFO); otherwise he takes another vacation and so on. The vacation model we consider here is the ungated system i.e. exhaustive system. It is shown here that the interarrival, service, vacation and server operation time can be cast with markov based representation then this class of vacation models can then be studied as matrix-product problem which belongs to a class of matrix analytic family- thereby allowing us to use result from Alfa (2003) to solve the resulting matrix product problem. Most importantly it is shown that using discrete time modelling approach to study some vacation model is more appropriate and makes the model much more algorithmically tractable.

Keywords: Batch Arrival, Discrete time model, matrix product problem, vacation queues.

1.0 Introduction

Vacation in queueing context mean the period the server is not attending to a particular targeted queue. The server may be under repair, attending to other queues or simply forced to stop serving customers in the particular queue. Vacation model have been used extensively to study various systems, such as polling and priority systems.

In a polling system, N queues are attended to by one server who attends to only one queue at a time. The server attends to one queue for a period of time based on some predefined rules and then proceeds to the next queue and so on.

Consider an arbitrary queue among the N queues. The customers in this particular queue view the server as being away on a vacation because they are not being attended to. This example of polling system is very common in computer systems where a processor has to attend to several queues of jobs.

Another example is road intersection control by traffic signals. At any given time one section of the road receives the green signal for service while the other section receives the red signal to stop service. The sections which receive the red signal are not receiving service and to them the server is on a vacation**.**

Priority queues are also sometimes studied as vacation queues. Consider a single server system with at least two classes of customers in which there is a priority for service. A low priority group of customers may keep receiving service until a higher priority customer arrives, after which the server may switch, depending on the predefined rules, to serving this higher priority customer. While the higher priority customer is receiving service the low priority group of customers sees the server as having gone on a vacation.

Several types of method have been used to study vacation model, ranging from embedded Markov chain to the classical transform approach. It is my hope to use the matrix-geometric method, set up in discrete time (Alfa 2003 [2]) to investigate the batch arrival discrete time queue with server vacation. This includes single server vacation queues with batch arrival, provided that the service, vacation and operational times can be represented by a Markov based model and the system is set up in discrete time.

Vacation models are classified into two categories:

1.1 Gated systems

In a gated system, as soon as the server returns from a vacation it places a gate behind the last waiting customer. It then begins to serve only the customers who are within the gate, based on some rules of how many or how long it could serve.

1.2 Ungated systems

In an ungated system the server only applies the rule of how many or how long it could serve. Under each of the classification above, we have further features, such as: single or multiple vacations; time-limited service – preemptive and non-preemptive, random interruptions for vacation, and others.

The subject of vacation queues has appeared in different literature in the last fifty years. For detail study of previous work in vacation models (Alfa 2003 [2]).

Here we will mention some few papers and books that are in the direction of this research work.

Choudhury (1997 [3]) uses the compound Poisson arrival and generalized vacation to analyze Batch Arrival Poisson queue with vacation.

In another development Shin and Pearce (1998 [13]) treat the batch Markovian arrival process whose vacation schedule and lengths of whose vacation times depend on the queue length of the system at the beginning of a vacation. Alfa (2003 [2]) provides a generalized form for a class of discrete time vacation model. He provides a unified framework for analyzing vacation models in discrete time and present matrix-analytic method for analyzing them.

Fiems *et al* (2004) investigate the gated multiple-vacation queue in discrete time. This generalized multiple vacation queueing model allows the capture of performance amongst, the multiple-vacation, the single-vacation and the limited multiple-vacation gated queueing systems.

Jau-Chuan (2004 [7]) studies the N policy $M^{[x]}/G/1$ queue with server vacations; startup and breakdowns, where the arrival form a compound poisson process and service times are generally distributed.

Ojobor (2006 [10]) study the effect of two queue discipline (FIFO and LIFO) on some measure of performance of a single server queue system using simulation. The approach is to generate arrival times and service for 200 customers and the customer is served through a single server queueing system under each queue discipline.

Other class of vacation model of interest to researcher lately is the case of vacation models in retrial systems. This class is very important when studying mobile communication and some computer networks. For results see Xiaoyong and Xiaowu (2007 [15]).

Modern telecommunication systems have become more digital systems than analog these days. It is therefore more appropriate to develop vacation models which are applicable to these systems using discrete time approach.

The aim of the current contribution is to investigate the Batch Arrival discrete time queue with server vacation. The goal is to model Batch Arrival Vacation models in discrete time and use the matrix analytic method to analyze the model. Here we restrict ourself to the ungated system-exhaustive case. In future work we shall look at the gated system.

2.0 Batch arrival discrete time Queue model with server vacation

In this section we shall consider the extension of Alfa (2003 [2]) model. The aim is to remodel Alfa model to allow room for batch arrival with server vacation. A rephrase of the model is given as follows:

- i = the number of items in the system.
- i = number of items inside a gate (when applicable).
- k = the phase of arrival: arrival is phase type with representation $(\boldsymbol{\alpha}, T)$. The arrival rate is λ. Here the arrival is in batches i.e. the batch arrival is phase type with representation $(\boldsymbol{\alpha}, T)$.
- j = the phase of service: service is phase type with representation $(\boldsymbol{\beta}, S)$.

- j = the phase of service interruption. This has m+1 phases including phase 0, when there is no interrupted service.
- l = the phase of vacation: vacation is phase type with representation $(\boldsymbol{\delta},L)$.
- u = the time clock of a server's visit (or the number served so far by a server during a visit) – the use depends on the model. This could also represent the phase of the operational time with representation (u,U).

The following parameters are also define

- N = the limited time of a single visit by a server for a time-limited service.
- M = the limited number of customers to be served during a server visit for number-limited service.

For some models where services may be interrupted we need to define a matrix Q which represents the phase at which an interrupted service begins when it resumed, given the interruption phase. The elements $Q_{j_1' j_1}$ refer to the probability that a service interrupted in phase j'_1 resumes in phase j_1 when the service re-starts. For example, a preemptive resume service has $Q = I$ and a preemptive repeat service rule has $Q = \mathbf{1}\boldsymbol{\beta}$.

2. 1 Exhaustive

We consider the ungated system i.e. exhaustive system. The state space for this system is described below:

- $\Delta_0^* = (0, k, 0), k = 1,2, \dots n$
- $\Delta_0^+ = (0, k, l), k = 1,2, \dots n, l = 1,2, \dots r,$
- $\Delta_i^v = (i, k, l), i = 1,2, \dots, k = 1,2, \dots, n, l = 1,2, \dots, r$
- $\Delta_i^s = (i, k, j), i = 1,2, \dots, k = 1,2, \dots, n, j = 1,2, \dots, m.$

For $\Delta_0^* = (0, k, 0)$, the first 0 refers to an empty system, and the second 0 refers to no vacation, i.e. the server is waiting at an empty queue for customers to arrive, after a vacation. For the case $\Delta_0^+ = (0, k, l)$, l refers to the phase of vacation. In both cases k refers to phase of arrival.

Let $\Delta_0 = \Delta_0^+ U \Delta_0^*$ and $\Delta_i = \Delta_i^v U \Delta_i^s$, where Δ_i^v is associated with the vacation states in which there are I customers in the system and Δ_i^s is associated with the service states with i customers in the system. The state space for the single vacation system is given by Δ as $\Delta = \Delta_0 \prod_{i=1}^{\infty} \Delta_i$. The state space for multiple vacation system is given by Δ as $\Delta = \Delta_0^+ \prod_{i=1}^{\infty} \Delta_i$. When the system is in states $\Delta_i, i \geq 1$, the chain makes transition to states $\Delta_{i-1}, \Delta_{i+1}$ or remain in state Δ_i. The transition probabilities are not level dependent, except for level i=1. When the system is in states Δ_0 the chain can only have transitions to states Δ_1 or remain in states Δ_0. This Markov chain is thus a level-independent QBD. If we now label the states in lexicographic order, and let the first index be $i, i \geq 0$, the resulting transition matrix for this Markov chain can then be represented in equation 2.1 below. We then apply the matrix geometric result to solve it.

The interior block matrices for the system, i.e. the matrices A_0, A_1, and A_0 which represent the transition from Δ_i to Δ_{i+1}, Δ_i to Δ_i and Δ_i to Δ_{i-1}, $\forall i \geq 2$

Here we shall reblock Alfa model to allow room for batch arrival. The block matrices for the system(other than the boundary ones) are given below:

$$p = \begin{bmatrix} B & C & & & & \\ E & A_1 & A_0 & & & \\ & A_2 & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & & & \Lambda \\ & & & & \mathrm{M} & \mathrm{M} \end{bmatrix} \quad (2.1)$$

where

$$A_0 = \begin{bmatrix} (T^0\alpha)\otimes L & (T^0\alpha)\otimes(L^0\beta) \\ 0 & (T^0\alpha)\otimes S \end{bmatrix}$$

$$A_1 = \begin{bmatrix} T\otimes L & T\otimes(L^0\beta) \\ 0 & T\otimes S + (T^0\alpha)\otimes S^0\beta \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & T\otimes(S^0\beta) \end{bmatrix}$$

Here the term $(T^0\alpha)\otimes L$ on the top left corner refers to transition during a vacation with batch arrival; the term $(T^0\alpha)\otimes(L^0\beta)$ on the top right corner refers to a transition from vacation to service commencement with batch arrival; and the term $(T^0\alpha)\otimes S$ on the bottom right corner refers to a transition during service with batch arrival. The term on the bottom left corner is zero because it refers to a transition from a service with batch arrival which is not possible since, if there is any customer in the system during service the server will not go on vacation at all, since A_0 refers to a transition with bulk arrival in the system. The meaning of the matrix blocks A_1 and A_2 are easily interpreted likewise.

Next we apply the matrix geometric solution by Neut (1984 [9]).

2.2 Matrix-geometric solutions for P

These are of finite order i.e. n,m,r, and $s < \infty$. The markov chain represented by p_1 is positive recurrent if $\pi A_0 \mathbf{1} < \pi A_2 \mathbf{1}$, where $\pi = \pi A, \pi\mathbf{1} = 1$ and $A=A_0+A_1+A_2$

For special cases it is possible to obtain simpler expressions for stability condition.

If the stability condition holds, we show how to obtain x. There exists a matrix R which is the minimal non-negative solution to the matrix quadratic equation, $R = A_0 + RA_1 + R^2A_2$. For a stable system the spectral radius R is less than 1. From matrix-geometric theorem we know that $x_{i+1} = x_iR = x_1R^i$, $i \geq 1$. If the vector $[y_0,y_1]$ is obtained as the eigenvector corresponding to the eigenvalue of 1 the matrix is given as

$$\begin{bmatrix} C & D \\ E & A_1 + RA_2 \end{bmatrix}$$

Further $y = Y_0^1 + Y_1 (I - R)^{-1}1$, then $x_0 = Y^{-1}y_0$ and $x_1 = Y^{-1}y_1$. We can therefore compute x using the matrix – geometric theorem, after which the performance measures can be obtained. In addition we talk about the matrix G which is the minimal non-negative solution to the matrix quadratic equation

$$G = A_2 + A_1G + A_0G^2.$$

The matrix G is stochastic if the Markov chain is positive recurrent. The matrix G is used in studying the busy period of the system. The relationship between R and G is $R = A_0(I - A_1 - A_0G)^{-1}$. Nearly all ungated systems can be developed as Markov chains with this type of QBD process.

3.0 Performance measures

We shall look at the performance of the system understudy. First we define f_i as a column vectors whose elements is either 0 or 1. We also define f_i^1 as row vector which is of the same order as x_i. The element of x_i have a relationship which corresponds to the element of f_i^1 .

Let Γ_i represent the set of all location in f_i^1 for which there are 1's, hence all the other location have 0's. Example, $\Gamma_i(i = \overline{i_1^1 \quad i_2^1})$ which implies the values of 1's in all location where the number in the gate is between i_1^1 *and* i_2^1 inclusive. Hence we write the corresponding f_i^1 vector as $f_i^1(\Gamma_i) = f_i^1(i^1 = \overline{i_1^1 \quad i_2^1})$.

3.1 Queue length distribution

Based on the result above, the vector x has been obtained. Our interest here is to obtain some key performance measures related to the queue length.

Let y_i be the marginal probability of finding i (Batch) customers in the system at an arbitrary time, then $y_i = x_i 1$. Denote C_n as the mean numbers of customers in the system at an arbitrary time.

$$C_n = x_i(1-R)^{-2}1$$

Let v_0 be the probability that the server is on vacation, then $v_0 = x_0 1 + x_1(1-R)^{-1} f_1(l = \overline{1,r})$,

Note here that we are assuming multiple vacation system.

Define y_i^v as the conditional probability that there are i customers waiting, when the server is on vacation.

That is, $y_i^v = \frac{x_i \quad f_i(l=\overline{1,r})}{v_o}$, $i \geq 1$ and $y_o^v = \frac{x_o 1}{v_o}$

The conditional expectation of the number of customers waiting when the server is on vacation c_v is given as

$$c_v = \frac{x_1(I-R)^{-2} f_1(l = \overline{1,r})}{v_o}$$

Denote the probability that service is ongoing at arbitrary time by s_0. Then we have $s_0 = 1 - v_0$.

Define y_i^s as the conditional probability that there are i customers waiting, when the server is busy serving, then, $y_i^s = \frac{x_i \quad (1-f_i(l=\overline{1,r}))}{s_o}$, $i \geq 1$

Let c_s be the conditional expectation of the number of customers waiting when the server is busy, then

$$c_s = x_1(I-R)^{-2} \frac{1 - f_1(l = \overline{1,r})}{s_o}$$

3.2 Distribution of work in the system

To write the equation for the work in the system in the level-independent case understudy, we need to define the following $s_1 = s^0\beta$, $G^{(0)}(0) = I$, $D_o = T$

$D_1 = T^o\alpha$, $x_i = [x_i^v, x_i^s]$, where v represent vacation and s service. Note also that $G^o(\omega) = 0\ \forall$ $\omega \geq 1$, $G^{(v)}(\omega) = 0 \ \ \omega < v$

The probability v_a^p of the amount of work in the system at the instant of return from a vacation is 'a' is given by

$$v_a^p = c^{-1}[x_o^v(D_1 \otimes L^o) \otimes G_a^{(1)} 1 + \sum_{i=1}^{a}\sum_{d=0}^{1} x_i^v((D_d \otimes L^o) \otimes \left(Q^* G^{(i+d)}(a)\right) 1)]$$

$$a \geq 1$$

where $c = \sum_{a=1}^{\infty}[x_o^v((D_1 \otimes L^o) \otimes G_a^{(1)})1 + \sum\sum x_i^v \left((D_d \otimes L^o) \otimes \left(Q^* G^{(i+d)}(a)\right)\right) 1]$

The probability v_a^v that the amount of work left behind just after vacation start 'a' is given by

$$v_o^v = b^{-1} \sum x_{1,u}^s (D_o \otimes s^o)1 \ and$$

$$v_a^v = b^{-1} \sum_{i=1}^{a} \sum_{w=1}^{a} \sum_{d=0}^{1} x_{i,N}^s (D_d \otimes ((s^{w-1} s^o) G^{(i+d-1)} (a-w+1)))1$$

where $b = \sum_{u=1}^{N} x_{1,u}^s (D_o \otimes s^o)1$

$$+ \sum_{a=1}^{\infty} \sum_{i=1}^{a} \sum_{w=1}^{a} \sum_{d=o}^{1} x_{i,N}^s (D_d \otimes ((s^{w-1} s^o) G^{(i+d-1)} (i-w+1)))1.$$

3.3 Waiting time distribution

Let $x_i, i \geq 0$ be partition as $x_i = [x_i^v, x_{i,1}^s, \ldots, x_{i,N}^s]$. Define the vectors z_i, $i \geq 0$ as the corresponding vectors to x_i, where z_i corresponds to the steady state vector of the system as seen by a (Batch) arriving customers. Then

$$z_0^v = \lambda^{-1} x_o^v((T^o \alpha) \otimes (L + L^o \delta)) + \lambda^{-1} \sum_{u=1}^{N} x_{1,u}^s ((T^o \alpha) \otimes (s^o \delta))$$

$$z_i^v = \lambda^{-1} \big[x_i^v((T^o \alpha) \otimes L \otimes I(m+1)) + x_{i,N}^s((T^o \alpha) \otimes \delta \otimes s^*) + x_{i+1,N}^s((T^o \alpha) \otimes \delta \otimes s^o) \big] \quad i \geq 1$$

$$z_{i,1}^s = \lambda^{-1} \big[x_i^v((T^o \alpha) \otimes (L^o \otimes Q^*)) \big]$$

$$z_{i,u}^s = \lambda^{-1} \big[x_{i,u-1}^s((T^o \alpha) \otimes s) + x_{i+1,u-1}^s((T^o \alpha) \otimes (s^o \beta)) \big]$$

$$i \geq 1, 2 \leq u \leq N$$

We define $\tilde{z}_o^o = z_o^v(1 \otimes I) and$ for $i \geq 1$ define $\tilde{z}_{i,o}^{v,o} = z_i^v(1 \otimes I)$

$$\tilde{z}_{i,u}^{s,o} = z_{i,u}^s(1 \otimes I)$$

Let $\tilde{z}_i^o = [\tilde{z}_{i,o}^{v,o}, \tilde{z}_{i,1}^{s,o}, \tilde{z}_{i,2}^{s,o}] \qquad i \geq 1$

Finally, let $\tilde{z}^o = [\tilde{z}_{o,}^o\ \tilde{z}_{1,}^o\ \tilde{z}_{3,}^o \ldots] \ also$ let $\tilde{z}^{n+1} = \tilde{z}^n \tilde{p}, \ \ n \geq 0$

$\tilde{z}^n$ is partition the same way as $\tilde{z}^0$.

Let W_a be the probability that a customer waiting time is less than or equal to a unit of time. Then

$$w_a = \tilde{z}_o^a 1 \quad a \geq 0$$

4.0 Conclusion

We have showed here that the matrices Presented by Alfa (2003 [2]) can be re-blocked to allow batch arrival and the method of matrix-product is then used to solve the resulting matrix problem.

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