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Relative controllability of neutral functional differential systems with nonlinear base

R. A. Umana
Department of Mathematics and Computer Science
Federal University of Technology, Owerri, Nigeria

Abstract

In this paper, sufficient conditions are established for the relative controllability of a special class of nonlinear neutral systems in which the base is strongly nonlinear and with time varying multiple delays in control. The results are established by using the Schauder fixed point theorem.

Keywords: Controllability, neutral functional differential system, nonlinear base, multiple delays, Schauder fixed point theorem.

1.0 Introduction

It is well known that the future state of realistic models in the natural sciences, economics and engineering depend not only on the present but on the past state and the derivative of the past state. Such models which contain past information are called hereditary systems. Neutral functional differential equations are characterized by a delay in the derivative. Equations of this type have applications in many areas of applied mathematics. There are simple examples from biology (predator-prey, Lotka-Volterra, spread of epidemics), from economics (dynamics of capital growth of global economy), and from engineering (mechanical and aerospace: aircraft stabilization and automatic steering using minimum fuel and effort, control of a high-speed closed air circuit wind tunnel; computer and electrical engineering: fluctuations of current in linear and nonlinear circuits, flip-flop circuits, lossless transmission lines), (see Chukwu [6]). A good guide concerning the literature for neutral functional differential equations is the Hale and Verduyn Lunel book [12] and the references therein. Some of the authors who have contributed to the study of the theory of neutral functional differential equations include Chukwu [5], Hernandez and Henriquez [13], Balachandran and Dauer [2], and Fu and Ezzinbi [10].

Controllability is the property of being able to steer between two arbitrary points in the state space using a set of admissible controls. The controllability of neutral systems has been studied by several authors including Balachandran and Anandhi [1], Balachandran and Sakthivel [3], Bouzahir [4], Umana [15], Fu [9] and Li et al [14].

The main purpose of this paper is to examine the relative controllability property of a special class of nonlinear neutral systems described by

$$\frac{d}{dt}D(t,x_t) = f\left(t,x_t,u(t)\right)+\sum_{i=0}^{N}B_i\left(t,x(t),u(t)\right)u\left(w_i(t)\right)$$

in which the base is strongly nonlinear. Our approach, similar to one used by Do [8] for nonlinear neutral systems, is to define the appropriate control and its corresponding solution by an integral equation. This equation is then solved by applying the well known Schauder fixed point theorem.

However, it should be stressed that the most literature in this direction has been mainly concerned with controllability problems for nonlinear perturbations of linear neutral functional differential systems in which the base system is inherently linear and controllable, and the perturbations are assumed to satisfy some growth conditions. In contrast to these studies, our current paper assumes that f be nonlinear.

e-mail: reubenandy@yahoo.com

2.0 Preliminaries

Suppose $h > 0$ is a given number, $E = (-\infty,\infty)$, E^n is a real n-dimensional linear vector space with norm $|.|$ and C = $C([-h,0] \to E^n)$ is the space of continuous functions from $[-h,0] \to E^n$ with sup norm. Let x be a function from $[\sigma - h, t_1] \to E^n$. Let $t \in [\sigma, t_1] \subset E$. We use the symbol x_t to denote the function on $[-h,0]$ defined by $x_t(s) = x(t + s)$ for $s \in [-h,0]$.

We consider systems of nonlinear functional differential equations of neutral type with time varying multiple delays in control having the form

$$\frac{d}{dt}D(t,x_t) = f\left(t,x_t,u(t)\right) + \sum_{i=0}^{N} B_i\left(t,x(t),u(t)\right)u\left(w_i(t)\right) \qquad (2.1)$$

where $x \in E^n$, u is an m-dimensional control function with $u \in C([\sigma,t_1],E^m)$, $B_i(t,x,u)$. $i = 0, 1, ..., N$ are $n \times m$ matrix functions, continuous in (t,x,u) and $f: E \times C \times E^m \to E^n$ is continuous and uniformly Lipschitzian in the last two arguments. The continuous strictly increasing functions $w_i(t):[\sigma,t_1] \to E$, $i = 0, 1, ..., N$, represent deviating arguments in the control, that is, $w_i(t) = t - h_i(t)$, where $h_i(t)$ are lumped time varying delays for $i = 0, 1, ..., N$. The operator D, $D: E \times C \to E^n$ is atomic at 0 and uniformly atomic at 0 in the sense of Hale [11]. Instead of the atomicity assumption on D, we may assume that D is of the form $D(t,\phi) = \phi(0) - g(t,\phi)$ where $g: E \times C \to E^n$ is continuous and is uniformly nonatomic at zero on $E \times C$ in the following sense:

***Definition* 2.1**

For any $(t,\phi) \in E \times C$, *and* $\mu \geq 0$, $s \geq 0$, *let*

$$Q(t,\phi,\mu,s) = \left\{\varphi \in C : (t,\varphi) \in E \times C, \|\varphi - \phi\| \leq \mu, \varphi(\theta) = \phi(\theta), \theta < -s, \theta \in [-h,0]\right\}.$$

We say that a continuous function $g: E \times C \to E^n$ is uniformly nonatomic at zero on $E \times C$ if, for any $(t,\phi) \in E \times C$, there exist $s_0 > 0$, $\mu_0 > 0$ independent of (t,ϕ), and a scalar function $\rho(t,\phi,\mu,s)$, defined and continuous for $(t,\phi) \in E \times C$, $0 \leq s \leq s_0$, $0 \leq \mu \leq \mu_0$, nondecreasing in μ, s such that $\rho_0 = \rho(E \times C, \mu_0, s_0) = \sup_{E \times C} \rho(t,\phi,\mu_0,s_0) < 1$ and $\left|g(t,\varphi) - g(t,\phi)\right| \leq \rho_0 \|\varphi - \phi\|$ for $t \in E$, $\varphi \in Q(t,\phi,\mu,s)$ and all $0 \leq s \leq s_0$, $0 \leq \mu \leq \mu_0$.

Definition 2.2

Given $\sigma \in E$, $\phi \in C$, *we say* $x(\sigma,\phi)$ *is a solution of* (2.1) *with initial value* ϕ *at* σ *if there exists* $a > 0$ *such that* $x \in C\left([\sigma - h, \sigma + a, E^n\right)$, $x_\sigma = \phi$ *and* $D(t,x_t)$ *is continuously differentiable on* $(\sigma, \sigma + a)$ *and satisfies* (2.1) *on* $(\sigma, \sigma + a)$.

It is known ([7] and the references therein) that under the prevailing assumptions on D, f, g, B_i and u for each $\phi \in C$ there is a unique solution of (2.1) with initial value ϕ at σ. The solution is continuous with respect to initial data and parameter u.

***Definition* 2.3**

The set $z(t) = \left\{x(t), x_t, u_t\right\}$ *is said to be the complete state of the system* (2.1) *at time* t.

***Definition* 2.4**

The system (2.1) is said to be relatively controllable on $[[\sigma, t_1]$ if for every initial complete state $z(\sigma)$ and every vector $x_1 \in E^n$, there exists a control $u \in C\left([\sigma,t_1],E^m\right)$ such that the solution $x(t) = x(t,\sigma,\phi,u)$ of (2.1) satisfies $x_\sigma(\cdot,\sigma,\phi,u) = \phi$, $x(t_1,\sigma,\phi,u) = x_1$.

A function x is a solution of (2.1) through (σ,ϕ) if and only if there exists a $\tau > 0$ such that x satisfies the equation $D(t,x_t) = D(\sigma,\phi) + \int_\sigma^t f(s,x_s,u(s))ds$

$$+\int_{\sigma}^{t} X(t,s,x,u)\sum_{i=0}^{N} B_i(s,x,u)u(w_i(s))ds, \quad t\in[\sigma,\tau] \tag{2.2}$$

$x_\sigma = \phi$ where $X(t,s,x,u)$ is an $n \times n$ matrix function defined for $\sigma \le s \le t+h$, continuous is s from

the right, of bounded variation in s, $X(t,s,x,u) = 0$, $t < s \le t + h$. Since $D(t,x_t) = x(t) - g(t,x_t)$. We deduce that the solution $x(t)$ of (2.1) is given by $x(t+\sigma) = \phi(t), \quad t \in [-h, 0]$,

$$x(t) = D(\sigma,\phi) + g(t,x_t) + \int_{\sigma}^{t} (s, x_s, u(s))ds$$

$$+\int_{\sigma}^{t} X(t,s,x,u)\sum_{i=0}^{N} B_i(s,x,u)u(w_i(s))ds, \quad t\in[\sigma,\tau],\ t \ge \sigma . \tag{2.3}$$

The function $w_i:[\sigma,t_1] \to E$, $i = 0, 1, ..., N$, are twice continuously differentiable and strictly increasing in $[\sigma,t_1]$. Furthermore, $w_i(t) \le t$ for t Є $[\sigma,t_1]$, $i = 0, 1, ..., N$. Let us introduce the following time lead functions r_i with $r_i(t):[w_i(\sigma), w_i(t_1)] \to [\sigma,t_1]$, $i = 0, 1, ..., N$, such that $r_i(w_i(t_1))$ for t Є $[\sigma,t_1]$. Without loss of generality, it can be assumed that $w_0(t) = t$ and the following inequalities hold for

$$t = t_1{:}h = w_N(t1) \le w_{N-1}(t_1) \le \ldots \le w_{m+1}(t_1) \le \sigma = w_m(t_1) < w_{m-1}(t_1) \le \ldots \le w_1(t_1) \le w_0(t_1) = t_1 . \tag{2.4}$$

Using the time lead function and the inequalities (2.4) we have $x(t+\sigma)=\phi(t), \quad t\in[-h,0]$

$$x(t_1) = D(\sigma,\phi) + g(t,x_t) + \int_{\sigma}^{t} (s,x_s,u(s))ds + \sum_{i=0}^{m}\int_{w_i(\sigma)}^{\sigma} X(t,s,x,u)B_i(r_i(s),x,u)\dot{r}_1(s)\eta(s)ds$$

$$+\sum_{i=m+1}^{N}\int_{w_i(\sigma)}^{w_i(t_1)} X(t,s,x,u)B_i(r_i(s),x,u)\dot{r}_1(s)\eta(s)ds + \sum_{i=0}^{m}\int_{(\sigma)}^{(t_1)} X(t,s,x,u)B_i(r_i(s),x,u)\dot{r}_1(s)u(s)ds \tag{2.5}$$

where $u(s) = \eta(s)$ for s Є $[\sigma - r,\sigma]$. Define, $p(t,x,u) = D(\sigma,\phi) + g(t,x_t) + \int_{\sigma}^{t} f(s,x_s,u(s))ds$,

$$q(t_1,x,\eta) = \sum_{i=0}^{m}\int_{w_i(\sigma)}^{\sigma} X(t,s,x,u)B_i(r_i(s),x,u)\dot{r}_1(s)\eta(s)ds + \sum_{i=m+1}^{N}\int_{w_i(\sigma)}^{w_i(t_1)} X(t,s,x,u)B_i(r_i(s),x,u)\dot{r}_1(s)\eta(s)ds$$

$$S(t,s,x,u) = \sum_{i=0}^{m} X(t,s,x,u)B_i(r_i(s),x,u)\dot{r}_1(s)$$

and the $n \times n$-dimensional controllability matrix $W(\sigma,t_1,x,u\int_{\sigma}^{t_1} S(t_1,s,x,u)S^*(t_1,s,x,u)ds$ where the star denotes matrix transpose. Then equation (2.5) becomes $x(t+\sigma)=\phi(t), \quad t\in[-h,0]$

$$x(t_1) = p(t,x,u) + q(t_1,x,\eta) += \int_{\sigma}^{t_1} S(t_1,s,x,u)u(s)ds \tag{2.6}$$

It is clear that x_1 can be obtained if there exist continuous x and u such that

$$u(t) = S^*(t_1,t,x,u)W^{-1}(\sigma,t_1,x,u)\left[x_1 - p(t_1,x,u) - q(t_1,x,\eta)\right] \tag{2.7}$$

and

$$x(t) = p(t,x,u) + q(t_1,x,\eta) += \int_{\sigma}^{t} S(t,s,x,u)u(s)ds \tag{2.8}$$

Now we will find conditions for the existence of such x and u. If $\alpha_i \in L^1[\sigma,t_1]$, $i = 1,2,...,q$, the $\|\alpha_i\|$ is the L^1 norm of $\alpha_i(s)$. That is, $\|\alpha_i\| = \int_{\sigma}^{t_1} |\alpha_i(s)|ds$.

3.0 Main Results

Theorem 3.1

In (2.1) *assume that*

(*i*) *the function $g:E \times C \to E^n$ is uniformly nonatomic at zero on $E \times C$ and $f: E \times C \times E^m \to E^n \times$ is continuous*;

(*ii*) the continuous function $F_i:E^n \times C^m \to E^+$ and L^1 functions $\alpha_i:E \to E^+$, $i = 1, 2, \ldots, q$ are such that

$$\left| f(t,\phi,u) \right| \le \sum_{i=1}^{q} \alpha_i(t) F_i(\phi,u) \text{ for every } (t,\phi,u) \in E \times C \times E^m,$$

where
$$\lim_{r\to\infty} \sup \left(r - \sum_{i=1}^{q} c_1 \sup \left\{ F_i(\phi,u) : \left\| (\phi,u \right\| \le r \right\} \right) = \infty .$$

Then system (2.1) is relatively controllable on $[\sigma,t_1]$ with $t_1 > \sigma + h$ if there exists a positive constant d such that for each pair of functions $(x,u) \in C[\sigma,t_1] \times E^m[\sigma,t_1]$ $\det W(\sigma,t_1,x,u) \ge d$.

Proof

Let $Q = C[\sigma,t_1] \times E^m[\sigma,t_1]$ and define the nonlinear continuous operator $T : Q \to Q$ by $T(x,u) = (y,v)$,

where
$$v(t) = S^*(t_1,t,x,u) W^{-1}(\sigma,t_1,x,u) \left[x_1 - p(t_1,x,u) - q(t_1,x,\eta) \right] \tag{3.1}$$

and
$$v(t) = p(t,x,u) + q(t,x,\eta) + = \int_{\sigma}^{t} S(t,s,x,u) v(s) ds \tag{3.2}$$

By our assumption, the operator *T* is continuous. Clearly the solutions *u* and *x* to (2.7) and (2.8) are fixed points of *T*. Our immediate aim now is to establish the existence of such fixed points by using the Schauder fixed point theorem. Indeed, let $F_i(r) = \sup \left\{ F_i(\phi,u) : \left\| (\phi,u) \right\| \le r \right\}$. Since the growth condition in (ii) is valid, there exists a constant $r_0 > 0$ such that $r_0 - \sum_{i=1}^{q} c_i F_i(r_0) \ge d$ or $\sum_{i=1}^{q} c_i F_i(r_0) + d \le r_0$ for some *d*. With this r_0, define $Q = Q(r_0)$. Now introduce the following notations

$$K = \max \left\{ \left\| X(t,s,x,u) \right\| : \sigma \le s \le t \le t_1 \right\},$$

$$k = \max_{\sigma \le s \le t_1} \left\{ S(t_1,s,x,u) t_1, 1 \right\},$$

$$a_i = 3k \max_{0 \le t \le t_1} \left\{ \left\| S^*(t_1,t,x,u) \right\| \left\| W^{-1}(\sigma,t_1,x,u) \right\| \left\| X(t_1,t,x,u) \right\| \left\| \alpha_i \right\| \right\},$$

$$b_i = 3K \left\| \alpha_i \right\|,$$

$$c_i = \max \left\{ a_i, b_i \right\},$$

$$d_1 = 3k \max_{0 \le t \le t_1} \left\| S^*(t_1,t,x,u) \right\| \left\| W^{-1}(\sigma,t_1,x,u) \right\| \left[\left| x_1 \right| + \left| p(t_1,x,u) \right| + \left| q(t_1,x,\eta) \right| \right],$$

$$d_2 = 3 \left[\left| p(t_1,x,u) \right| + \left| q(t_1,x,\eta) \right| \right],$$

$$d = \max \left\{ d_1, d_2 \right\}.$$

Now let $Q_{r_0} = \left\{ (x,u) \in Q : \left\| (x,u) \right\| \le r_0 \right\}$. If $(x,u) \in Q_{r_0}$, then from (3.1) and (3.2) we have

$$\left\| v \right\| \le \left\| S^*(t_1,t,x,u) \right\| \left\| W^{-1}(\sigma,t_1,x,u) \right\| \left[\left| x_1 \right| + \left| p(t_1,x,u) \right| + \left| q(t_1,x,\eta) \right| \right.$$

$$+\|X(t,s,x,u)\|\sum_{i=1}^{q}\alpha_i(s)F_i(x_s,u(s))ds] \le \frac{d_1}{3k}+\sum_{i=1}^{q}\frac{1}{3k}\alpha_i F_i(r_0) \le \frac{1}{3k}\left(d+\sum_{i=1}^{q}c_i F_i(r_0)\right) \le \frac{1}{3k}r_0 \le \frac{r_0}{3}$$

and $\|y\| \le |p(t,x,u)|+|q(t_1,x,\eta)|+\int_\sigma^t \|S(t,s,x,u)\|\|v\|ds+\int_\sigma^t \|X(t,s,x,u)\|\sum_{i=1}^{q}\alpha_i(s)F_i(x_s,u(s))ds$

$$\le \frac{d}{3}+k\|v\|+K\sum_{i=1}^{q}\|\alpha_i\|F_i(r_0) \le \frac{d}{3}+k\|v\|+\sum_{i=1}^{q}\frac{1}{3}c_i F_i(r_0) \le \frac{1}{3}\left(d+\sum_{i=1}^{q}c_i F_i(r_0)\right)+k\|v\| \le \frac{r_0}{3}+\frac{r_0}{3}=2\left(\frac{r_0}{3}\right)$$

Hence T maps Q_{r_0} into itself. Further, it is easy to see that $T(Q_r)$ is equicontinuous for all $r > 0$ [8]. By the Ascoli–Arzela theorem, $T(Q_{r_0})$ is compact in Q. Since Q_{r_0} is closed, bounded and convex, the Schauder fixed point theorem guarantees that T has a fixed point $(x,u)\in Q_{r_0}$ such that $T(x,u)=(x,u)$. Hence, for $(x,u)=(y,v)$, we have $x(t)=p(t,x,u)+q(t,x,h)+=\int_\sigma^t S(t,s,x,u)u(s)ds$. Thus the solutions of (2.7) and (2.8) exist. Hence the system (2.1) is relatively controllable on $[\sigma,t_1]$ with $t_1>\sigma+h$.

Inspired by the above ideas we have the following corollaries.

***Corollary* 3.1**

For the system (2.1) assume

(i) *condition* (*i*) *of Theorem* 3.1 *holds,*

(ii) $$\lim_{\|(\phi,u)\|\to\infty}\frac{|f(t,\phi,u)|}{\|(\phi,u)\|}=0 \tag{3.3}$$

uniformly in t, $t\in[\sigma,t_1]$. *Then* (2.1) *is relatively controllable on* $[\sigma,t_1]$ *with* $t_1>\sigma+h$ *if there exists a positive constant* d *such that for each pair of functions* $(x,u)\in C[\sigma,t_1]\times E^m[\sigma,t_1]$

$\det W(\sigma,t_1,x,u)\ge d$.

Proof

Let $F(\phi,u)=\sup\{|f(t,\phi,u)|: t\in[\sigma,t_1]\}$. Then

$$\lim_{r\to\infty}\left(r-\sum_{i=1}^{q}c_1\sup\{F_i(\phi,u):|(\phi,u)|\le r\}\right)=\infty . \tag{3.4}$$

if $$\liminf_{r\to\infty}\left(\frac{1}{r}\right)\sup\left\{F_i(\phi,u):|(\phi,u)|\le\frac{1}{c_1}\right\}\Bigg). \tag{3.5}$$

But condition (3.3) implies (3.5) by a modification of an argument of Do [8]. The required modification is the proof that if the corresponding sequence $\{(\phi_i,u_i\}$ is bounded we can assume it compact. Therefore (3.4) is valid and Theorem 3.1 can be concluded. Recall that $f(t,\phi,u)$ is said to be locally bounded in u if for any $M>0$, there is an $L>0$ such that $\|f(t,\phi,u)\|\le L$ for all $(t,\phi)\in E\times C$ and for all $\|u\|\le M$.

***Corollary* 3.2**

For the system (2.1) assume

(i) condition (i) of Theorem 3.1 holds,

(ii) $f: E\times C\times E^m$ *is locally bounded in* u *,*

(iii) $$\lim_{\|u\|\to\infty}\frac{\|f(t,\phi,u)\|}{\|u\|}=0 \tag{3.6}$$

uniformly in $(t,\phi)\in E\times C$. *Then (2.1) is relatively controllable on* $[\sigma,t_1]$, $t_1>\sigma+h$ *if there exists a positive constant* d *such that for each pair of functions* $(x,u)\in C[\sigma,t_1]\times E^m[\sigma,t_1]$ $\det W(\sigma,t_1,x,u)\geq d$.

Proof

Let $F(\phi,u)=\sup\{\|f(t,\phi,u)\|:t\in[\sigma,t_1]\}$. Then

$$\|f(t,\phi,u)\|\leq F(\phi,u) \text{ for every } (t,\phi,u)\in[\sigma,t_1]\times C\times E^m.$$

Because of (3.6) the following is valid:

$$\lim_{r\to\infty}\left(\frac{1}{r}\right)\sup\{F(\phi,u):\|(\phi,u)\|\leq r\})=0$$

As a consequence (3.5) holds, and the result follows.

4.0 Conclusion

In the paper, sufficient conditions for the relative controllability of neutral systems in which the base is strongly nonlinear and with time varying multiple delays in control have been formulated and proved. The approach used here was to define the appropriate control and its corresponding solution by an integral equation. This equation was then solved using the well known Schauder fixed point theorem.

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