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Solitons of flexual deformation of the basilar membrane

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Abstract

Solitons are very stable solitary waves in a solution of the plate equation obtain from the three dimensional equations of the linearized director theory for an elastic body (the basilar membrane) which were solve and solution shows solitons. We show that the solitary waves behave like particle. When there are located mutually far apart, each of them is approximately a travelling wave with constant shape and velocity. The stability of the solitons stems from the delicate balance of "nonlinearity" and "dispersion" in the model equations. Nonlinearity drives a solitary wave to concentrate further; dispersion is the effects to spread such a localized wave. In this exhibition, we present equation of soliton phenomena along with soliton solution.

1.0 Introduction

Solitons are nonlinear waves which is considered as solitary travelling wave pulse, solution of nonlinear partial differential equation (PDE). The equation describing the (unidirectional) propagation of wave on the surface of shallow channel was derived by Green & Niaghdi. Therefore, considering the basilar membrane as an elastic plate displacement in a continuum can conveniently make the study of the motion of both the basilar membrane and the fluid reasonable.

The basilar membrane can be considered as a thin plate, which is light and taut at the basal end of the cochlea and thick and loose near its apex in a medium (fluid-endolymph). It is elastic in nature so that under this fluid motion, it undergoes some form of deformation, which we shall consider as motion. Unlike the well-known theories of plates with loads, our load is not purely a load but a kind of solitary travelling wave load caused by the motion of the endolymph as the stapes rock the foot on the cochlea. We shall recall the mechanism of hearing such that when the stapes rock foot on the cochlea, it sets up wave motion in the endolymph in which the basilar membrane is immersed. In the absence of any sound or noise, the basilar membrane will be lying in the fluid with a load (i.e. the fluid thrust on it), equally distributed throughout its length.

We note that the basilar membrane in the fluid has hair cells on its surface. These hair cells on their own undergo some form of motion as the wave is generated in the endolymph, which causes the deformation of the basilar membrane. Since the motion of the hair cells is dependent on the motion of the basilar membrane, we then have to consider the motion of the hair cells along side with that of the basilar membrane.

Models which attempt to incorporate the properties of the basilar membrane as a plate are those of Chadwick, Inselberg and Steele [4]. The model of Chadwick and Inselberg [4] is two dimensional, so the plate is replaced by a beam. The exponential variation of compliance is based on shallow water wave theory. Steele's model is probably the most detailed yet attempted. It is a three dimensional model in which the basilar membrane appears as a plate. Various boundary conditions for the support of the plate at its edges are considered. A class of the cochlea models of direct concern in the present paper, may be described as follows:

(1) The model is an enclosed two-dimensional cavity and the basilar membrane appears on it as a thin plate which is light and taut at the basal end of the cochlea and thick and loose near its apex in a medium (fluid-endolymph).

(2) The flexural deformation of the basilar membrane is derived from the linear theories of the elastic plate.

(3) The simplest mathematical model, which contains the linear short time scale aspects of cochlea behaviour, is considered.

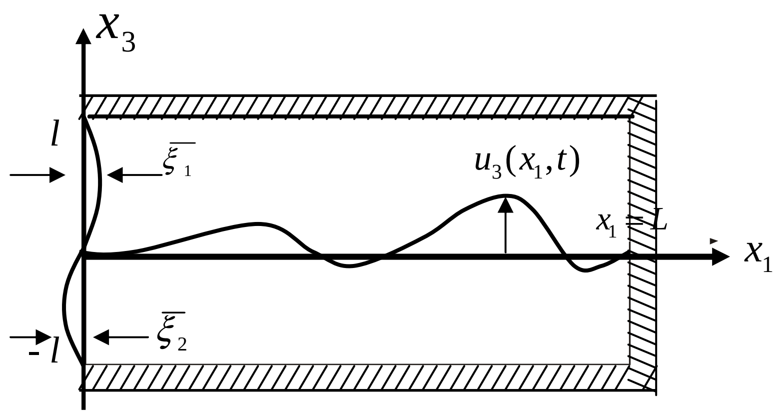

Figure 1.1

1.1 Elastic plate

The flexural deformations of the basilar membrane are derived from the linear theories of the elastic plate. The plate equations are obtained from the three dimensional equations of the linearized director theory for an elastic body. The equations resulting from the later derivation completely agree with those obtained previously for an elastic Cosserat plate. We neglect the equation for the directors.

The equation of such a plate has been given by Green and Naghdi [10], Tmoshenko and Woinowsky-Krieger, [26] and can be extracted in the form:

$$\alpha_3 \bar{u}_{3,\alpha\alpha} + \rho F_3 = \rho \ddot{\bar{u}}_3 \tag{1.1}$$

where α_3 is the flexural rigidity, ρ mass density of the basilar membrane. $(\ )_{,\alpha\alpha}$ is the Laplacian of the quantity ( ). As a model of the basilar membrane, on $x_3 = 0$, $0 < x_3 < L$ (see figure 1) the equation of flexural vibration of the basilar membrane is:

$$\alpha_3 \nabla^2 \bar{u}_3 - \rho \ddot{\bar{u}}_3 = \bar{p}_2(x_1, 0, t) - \bar{p}_1(x_1, 0, t) \tag{1.2}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x_1^2}$ and $-F_3 = \bar{p}_2(x_1,0,t) - \bar{p}_1(x_1,0,t)$ is the load on the basilar membrane. The boundary condition on the basilar membrane is given by:

$$\frac{\partial \bar{u}_3}{\partial t} = \frac{\partial \bar{\phi}_1}{\partial x_3}, \quad \frac{\partial \bar{u}_3}{\partial t} = \frac{\partial \bar{\phi}_2}{\partial x_3} \tag{1.3}$$

We seek a solution such that the field variables will be proportional to $e^{i\omega t} = e^{st}$.

We write $\phi = \mathrm{Re}(\bar{\phi} e^{st})$, $u_3 = \mathrm{Re}(\bar{u}_3 e^{st})$, $p_i = \mathrm{Re}(\bar{p}_i e^{st})$. The equations of motion of pressure variation now become

$$p_1 + \rho s \phi_1 = 0, \quad p_2 + \rho s \phi_2 = 0 \tag{1.4}$$

On $x_3 = 0, \quad 0 < x_1 < L$

$$\alpha_3 \nabla_{(1)}^2 u_3 - \rho s^2 u_3 = p_2(x_1, 0,) - p_1(x_1, 0,) \tag{1.5}$$

The boundary conditions (1.3) become

$$s u_3 = \frac{\partial \phi_1}{\partial x_3}, \quad s u_3 = \frac{\partial \phi_2}{\partial x_3} \tag{1.6}$$

at the boundary, $x_3 = 0$, for $x_1 = L$, $\frac{\partial \phi_1}{\partial x_1} = \frac{\partial \phi_2}{\partial x_1} = 0$ (1.7)

The outer walls are rigid and the basilar membrane is an elastic plate where undisturbed position is the plane $x_3 = 0$, which is also a plane of symmetry. The spaces above and below the basilar membrane are filled with a non-viscous, incompressible fluid. The basilar membrane is light and taut at the basal end of the cochlea, it is thick and loose near its apex.

We assume the end $x_1 = 0$ is under stress and $x_1 = L$ is stress free and as such

$$\frac{\partial \bar{u}_3}{\partial x_1}\Big|_{x_1 = L} = 0 \tag{1.8}$$

From equation (1.5), we obtain

$$\frac{\partial^2 \bar{u}_3}{\partial x_1^2} - \frac{\rho s^2 \bar{u}_3}{\alpha_3} = \frac{1}{\alpha_3}(\bar{\rho}_2 - \bar{\rho}_1) \tag{1.9}$$

From equation (1.4), and substituting values of ϕ_1 and ϕ_2, (from Adagba unpublished thesis, we obtain

$$\bar{\rho}_2 - \bar{\rho}_1 = \rho s\left(\bar{\phi}_1 - \bar{\phi}_2\right) = \frac{\rho s\left[\beta \cos\lambda\left(L - x_1\right)\cos h\lambda\left(1 - x_3\right) - \gamma\cos\lambda\left(L - x_1\right)\cos h\lambda\left(1 + x_3\right)\right]}{\cos\lambda L \cos h\lambda 1}$$

so that on $x_3 = 0,$ we obtain

$$\bar{\rho}_2 - \bar{\rho}_1 = \frac{\rho s\left(\beta - \gamma\right)\cos\lambda\left(L - x_1\right)}{\cos\lambda L} \tag{1.10}$$

substituting (1.10) into (1.9), gives

$$\frac{\partial^2 \bar{u}_3}{\partial x_1^2} - \frac{\rho s^2 \bar{u}_3}{\alpha_3} = \frac{\rho s(\beta - \gamma)\cos\lambda(L - x_1)}{\alpha_3 \cos\lambda L} \tag{1.11}$$

The solution of the homogenous part is $\bar{u}_3 = A\cosh\lambda_0 x_1 + B\sinh\lambda_0 x_1$ (1.12)

where

$$\lambda_0 = \sqrt{\frac{\rho s^2}{\alpha_3}} = i\omega\sqrt{\frac{\rho}{\alpha_3}} \tag{1.13}$$

see Titchmarch, Barrett and Wylie [27]. Let the particular integral be

$$\bar{u}_3 = k\cos\lambda(L - x_1) \tag{1.14}$$

Substituting (1.14) into (1.11), gives

$$-k\lambda^2\cos\lambda\left(L - x_1\right) - \frac{\rho s^2 k\cos\lambda\left(L - x_1\right)}{\alpha_3} = \frac{\rho s\left(\beta - \gamma\right)\cos\lambda\left(|L - x_1\right)}{\alpha_3\cos\lambda L}$$

which upon simplification, gives

$$k = \frac{-\rho s(\beta - \gamma)}{(\lambda^2\alpha_3 + \rho s^2)\cos\lambda L} \tag{1.15}$$

valid for $x_3 = 0$. Substituting (1.15) into (1.14), we obtain the particular integral

$$\bar{u}_{3\rho} = \frac{\rho s(\beta - \gamma)\cos\lambda(L - x_1)}{(\lambda^2\alpha_3 + \rho s^2)\cos\lambda L} \tag{1.16}$$

Hence, the complete solution is

$$\bar{u}_3\left(x_1\right) = A\cos h\lambda_0 x_1 + B\sin h\lambda_0 x_1 - \frac{\rho s\left(\beta - \gamma\right)\cos\lambda\left(L - x_1\right)}{\left(\lambda^2\alpha_3 + \rho s^2\right)\cos\lambda L} \tag{1.17}$$

Since the apex is assumed loose $\frac{\partial u_3}{\partial x_1}\Big|_{x_1=L}=0$ is satisfied identically and it implies that $A\sin h\lambda_o L + B\cos h\lambda_o L = 0,$ which gives

$$B = \frac{A\sinh\lambda_0 L}{\cosh\lambda_0 L} \tag{1.18}$$

Substituting (1.18) into (1.17), we obtain

$$\bar{u}_3(x_1) = \frac{A\cosh\lambda_0(L-x_1)}{\cosh\lambda_0 L} - \frac{\rho s(\beta-\gamma)\cos\lambda(L-x_1)}{(\lambda^2\alpha_3+\rho s^2)\cos\lambda L} \tag{1.19}$$

Applying the last equality in equation (1.7), (the values ϕ_1 and ϕ_2, see 10) $\frac{\partial\phi_1}{\partial x_3}=\frac{\partial\phi_2}{\partial x_3}$ on $x_3=0$ yields $\frac{-\beta\cos\lambda\left(L-x_1\right)\lambda\sinh\lambda l}{cos\lambda L\cosh\lambda l}=\frac{\gamma\cos\lambda\left(L-x_1\right)\lambda\sinh\lambda l}{\cos\lambda L\cosh\lambda l}$. Upon simplification leads to the result

$$-\beta=\gamma \tag{1.20}$$

Substituting (1.20) into (1.19), gives

$$\bar{u}_3(x_1) = \frac{A\cosh\lambda_0(L-x_1)}{\cosh\lambda_0 L} - \frac{2\beta\rho s\cos\lambda(L-x_1)}{(\lambda^2\alpha_3+\rho s^2)\cos\lambda L} \tag{1.21}$$

Also, the first equality of the same (1.16) gives $s\bar{u}_3=\frac{\partial\phi_1}{\partial x_3}\Big|_{x_3=0}$ which implies that

$$\frac{A\cosh\lambda_0(L-x_1)}{\cosh\lambda_0 L} - \frac{2\beta\rho s^2\cos\lambda(L-x_1)}{(\lambda^2\alpha_3+\rho s^2)\cos\lambda L} = -\frac{-\beta\lambda\tanh\lambda l\cos\lambda(L-x_1)}{\cos\lambda L} \tag{1.22}$$

Substituting equation (1.13) into (1.22), yields

$$\frac{sA\cosh i\omega\sqrt{\frac{\rho}{\alpha_3}}\left(L-x_1\right)}{\cosh i\omega\sqrt{\frac{\rho}{\alpha_3}}L} = \frac{\left\{2\rho s^2-\left(\lambda^2\alpha_3+\rho s^2\right)\lambda\tanh\lambda l\right\}\beta\cos\lambda\left(L-x_1\right)}{\left(\lambda^2\alpha_3+\rho s^2\right)\cos\lambda L} \tag{1.23}$$

Since $\cos hi\omega\sqrt{\frac{\rho}{\alpha_3}}=\cos\omega\sqrt{\frac{\rho}{\alpha_3}}$, then

$$\frac{sA\cos\omega\sqrt{\frac{\rho}{\alpha_3}}\left(L-x_1\right)}{\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L} = \frac{\left\{2\rho s^2-\left(\lambda^2\alpha_3+\rho s^2\right)\lambda\tanh\lambda l\right\}\beta\cos\lambda\left(L-x_1\right)}{\left(\lambda^2\alpha_3+\rho s^2\right)\cos\lambda L}.$$

We can determine the constant A, if we take $\lambda=\omega\sqrt{\frac{\rho}{\alpha_3}},$ Hence, we have

$$A = \frac{2\rho s^2-\left(\lambda^2\alpha_3+\rho s^2\right)\beta\lambda\tan\lambda l}{s(\lambda^2\alpha_3+\rho s^2)} \tag{1.24}$$

Substituting (1.24) into (1.21) and simplifying, gives

$$u_3(x_1,t) = \frac{-\beta\sqrt{\frac{\rho}{\alpha_3}} thanh \omega\sqrt{\frac{\rho}{\alpha_3}} l \cos\omega\sqrt{\frac{\rho}{\alpha_3}}(L-x_1)}{s\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L} \tag{1.25}$$

Recalling that $u_3 = \mathrm{Re}\{\bar{u}_3 e^{st}\}$, then

$$u_3(x_1,t) = \frac{\mathrm{Re}\left\{-\beta\omega\sqrt{\frac{\rho}{\alpha_3}}\tanh\omega\sqrt{\frac{\rho}{\alpha_3}} l\cos\omega\sqrt{\frac{\rho}{\alpha_3}}(L-x_1)e^{st}\right\}}{s\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L}$$

$$= \frac{-\mathrm{Re}\left\{\beta\omega\sqrt{\frac{\rho}{\alpha_3}}\tanh\omega\sqrt{\frac{\rho}{\alpha_3}} l\cos\omega\sqrt{\frac{\rho}{\alpha_3}}(L-x_1)(\cos\omega t + i\sin\omega t)\right\}}{iw\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L}$$

Therefore

$$u_3(x_1,t) = \frac{-\beta\omega\sqrt{\frac{\rho}{\alpha_3}}\tanh\omega\sqrt{\frac{\rho}{\alpha_3}} l\sin\omega t\cos\omega\sqrt{\frac{\rho}{\alpha_3}}(L-x_1)}{\omega\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L} \tag{1.26}$$

$$= -\beta\omega\sqrt{\frac{\rho}{\alpha_3}}\tanh\omega\sqrt{\frac{\rho}{\alpha_3}} l \frac{\left(\sin\left(\omega t + \omega\sqrt{\frac{\rho}{\alpha_3}}(L-x_1)\right) - \sin\left(\omega t - \omega\sqrt{\frac{\rho}{\alpha_3}}(L-x_1)\right)\right)}{2\omega\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L}$$

$$= -\beta\sqrt{\frac{\rho}{\alpha_3}}\tanh\omega\sqrt{\frac{\rho}{\alpha_3}} l \frac{\left(\sin\omega\left(t - x_1\sqrt{\frac{\rho}{\alpha_3}} + L\sqrt{\frac{\rho}{\alpha_3}}\right) - \sin\omega\left(t + \sqrt{\frac{\rho}{\alpha_3}}x_1 - L\sqrt{\frac{\rho}{\alpha_3}}\right)\right)}{2\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L} \tag{1.27}$$

This is oscillatory with two travelling waves, one in the positive direction whereas the other in the negative direction with the same speed, see Jeffrey and Jeffrey [14], Mikilin, Murrray [19].

Consider the deflection given by (1.25), the maximum deflection with respect to the distance is at the helicotrema $(x_1 = L)$ as required by the place principle at low frequencies, see Furness and Hackey [6], Furukawa and Matura [7]. Hence (1.25) becomes

$$u_3(x_1) = \frac{-\beta\sqrt{\frac{\rho}{\alpha_3}}\tanh\omega\sqrt{\frac{\rho}{\alpha_3}}l\sin\omega t}{\cos\omega\sqrt{\frac{\rho}{\alpha_3}}L} \qquad (1.28)$$

This is actually the case for the real basilar membrane. The taper in the basilar membrane is needed for optimal low frequency behaviour, see Huggin and Licklider [11], Huxley, Inselberg and Forster [12].

A uniform basilar membrane provides excellent high frequency response for a basilar membrane modelled as a beam with uniform geometrical and elastic properties.

The high tones have oscillation patterns in which the maximum activity occurs in the basal region, while the low tones have broader patterns. The region of most vigorous action shifts toward the basal end as the frequency of the external signal increases. The phase of Hair cells response is synchronized with the basilar membrane. The HC at the apex behaves as might be expected, with a maximum excitation when the basilar membrane is between maximum displacement and velocity toward scala vestibuli. However, the neural recordings at the base indicate excitation with velocity toward scala tympan, see Johnstone and Yates 15], King [16], Pain [21], Peskin [22].

2.0 Conclusion

Solitons are very stable solitary waves in a solution of this equation. As the term “Soliton” suggests, this solitary waves behave like “particles”. When there are located mutually far apart, each of them is approximately a travelling wave with constant shape and velocity. The stability of the solitons stems from the delicate balance of “nonlinearity” and “dispassion” in the model equations. Nonlinearity drives a solitary wave to concentrate further, dispersion is the effects to spread such a localized wave. If one of these two competing effect is lost, solutions become unsuitable, and eventually, case to exist. In this respect, solitons are completely different “Linear waves” like sinusoidal waves. Infact, sinusoidal waves are rather unsuitable in some model equation of solitons phenomena.

Figure 2.1, graphically represents the deformation of the basilar membrane. This results in several solitary waves that interact and are propagated in a complex waveform. The shape of the graph depicts the flexural deformation of the basilar membrane as indicated by the model. The complex waveforms are determined by the fact that sound waves of common phase reinforce each other whereas, sound waves of opposite phase cancel when they meet.

Figure 2.2 is also the deformation the basilar membrane without the time function (t). It is only a function of x_1, and it shows solitary wave cycles slightly different from figure 2.1.

2.1 Deformation of the basilar membrane

α_3: = 2, L: 35, ρ: = 1, m_0: = 0.05, ω: = 10^4, *s*: = 10^4, 1: = 1, β: = 1, *N*: = 25, *i*: = 0, …, *N*, *j*: = 0, …, *N*, x_i: = 1.5 + 0.5*i*, t_j: = 1.5 + 0.15$_j$. $u_3(x,t) := \frac{-\left[\beta\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}\right]\tanh\left[\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}\cdot 1\right]\sin(\varpi,t)\cos\left[\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}(L-x)\right]}{\cos\left[\left[\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}\right]L\right]}$

$M_{ij} : u_3(x_i, t_j)$

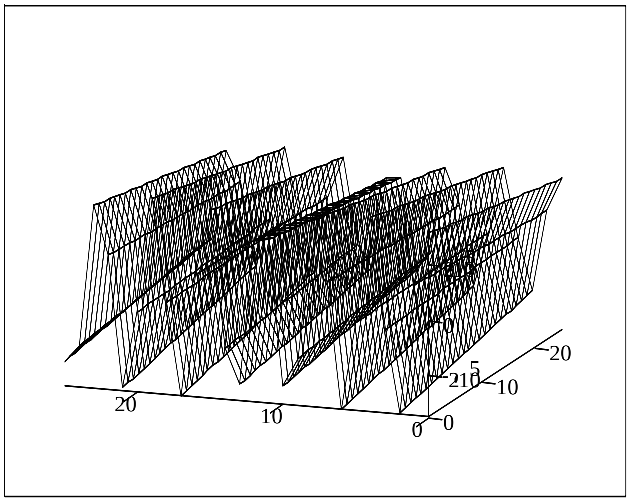

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Figure 2.1: Deformation of the basilar membrane.

2.2 Deformation of the basilar membrane along the x-axis

α_3: = 2, L: 35, ρ: = 1, m_0: = 0.05, ω: = 10^4, *s*: = 10^4, 1: = 1, β: = 1, *N*: = 25, *i*: = 0, …, *N*, *j*: = 0, …, *N*, x_i: = 1.5 + 0.5*i*, t_j: = 1.5 + 0.15$_j$. $u_3(x,t) := \dfrac{-\left[\beta\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}\right]\tanh\left[\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}\cdot 1\right]\cos\left[\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}(L-x)\right]}{\omega\cos\left[\omega\left(\frac{\rho}{\alpha_3}\right)^{0.5}\cdot L\right]}$

$M_{ij} : u_3(x_i, t_j)$

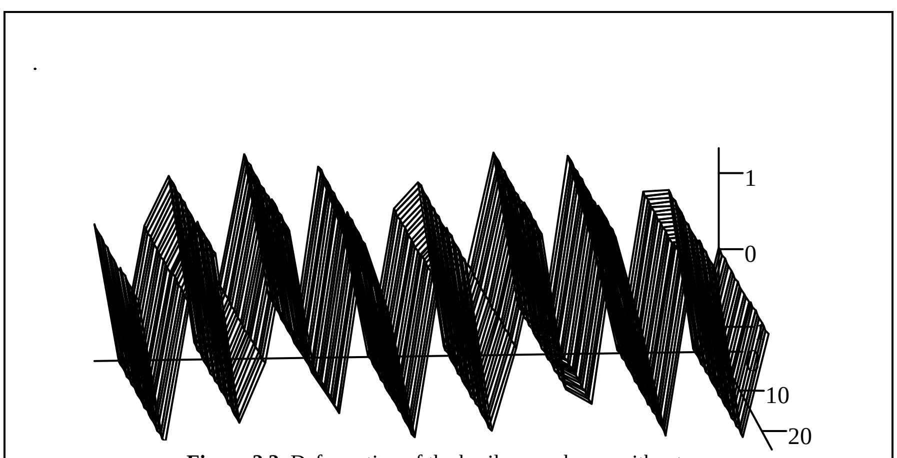

Figure 2.2: Deformation of the basilar membrane without the time function

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