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Mathematical model for flow around two-dimensional aerofoils at small angles of attack

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Abstract

The sole aim of this work is to develop a mathematical model for calculating certain parameters like velocity, angle of incidence, lift coefficient and geometric characteristics like maximum thickness and camber for both same and related aerofoils using Prandtl-Glauert* [1, 2] *and Gothert* [3] *similarity rules. The model is extended to include pressure at any point on the aerofoil profile as well as other geometric characteristics like chord, wing area and span. Applying the model to a numerical example, based on flight problem, these parameters are determined in compressible flow and compared with those in incompressible flow. Furthermore, the model which includes ratios of geometric characteristics of aerofoil like thickness/chord ratio and camber/chord ratio is extended to include wing area/span ratio, and these parameters are also determined in the example and compared for both same and related aerofoil.

Keywords: Mathematical model, two-dimensional aerofoils, similarity rules, small angles of attack.

1.0 Introduction

Aerodynamics is a branch of fluid dynamics that is concerned with the study of gas flow. On the other hand, an aerofoil is a streamlined body designed to produce lift when placed in a fluid stream, e.g. the wing section of an aircraft, Massey [4]. Various investigations have been done in the area of aerodynamics. For instance, Elcrat and Bassanini [5], in their study of free streamline boundary layer analysis for separated flow over an aerofoil, pointed out that the steady flow past a lifting surface at high Raynolds numbers and modest angle of attack can be thought of as consisting of two parts, the exterior flow in which the flow is essentially invisicid and a thin region near the body in which viscosity is important, the boundary layer. By combining boundary layer computation with a free streamline potential flow they obtained pressure distribution on an aerofoil section in which partial separation has occurred. The method proposed worked for angles of attack up to stall. Other contributors to the discussion of flows around aerofoils include, notably, Euler [6], Woods [7], Thomson and Bradley [8], Turkyilmazoglu [9] Hafez and Wahba [10], Hafez et al. [11], Ansari et al. [12], Wald et al. [13], Winckelmans et al. [14], Peters [15], etc.

In this work we shall use mathematical model to determine certain properties of aerofoils at small angles of incidence in compressible flow and compare them in incompressible flow. We shall also determine geometric characteristics and compare them for both same and related aerofoils. In doing this we assume the flow to be two-dimensional and also potential.

2.0 Mathematical Model

The aerodynamic performance in compressible flow may be related to that in the incompressible flow by the factor (see Prandtl [1] and Glauert [2])

$$\lambda = \left(1 - Ma^2\right)^{-\frac{1}{2}} \tag{2.1}$$

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where Ma is the Mach number of the free stream and λ a factor. Throughout, we shall use the suffix '*c*' to denote compressible flow while the suffix '*i*' denotes incompressible flow.

2.1 Same aerofoil

For the same aerofoil, if u_i is the velocity at any point on its profile in an incompressible flow and the velocity at the same point in the compressible flow is u_c, then by Prandtl – Glauert similarity rule [1, 2].

$$\frac{u_c}{u_i} = \left(1 - Ma^2\right)^{-\frac{1}{2}} = \lambda$$

or

$$u_c = \lambda u_i \tag{2.2}$$

Similarly, $\frac{(C_L)_c}{(C_L)_i} = \left(1 - Ma^2\right)^{-\frac{1}{2}} = \lambda$ or $(C_L)_c = \lambda (C_L)_i$ (2.3)

for lift coefficient C_L, and $\frac{\alpha_c}{\alpha_i} = \left(1 - Ma^2\right)^{-\frac{1}{2}} = \lambda$, or $\alpha_c = \lambda \alpha_i$ (2.4)

for angle of incidence α. Extending this rule to include pressure p, we find

$$\frac{p_c}{p_i} = \left(1 - Ma^2\right)^{-\frac{1}{2}} = \lambda \text{ or } p_c = \lambda p_i \tag{2.5}$$

2.2 Geometric characteristics of two (or related) aerofoils

For two aerofoils, one in an incompressible flow (suffix i) and the other in the compressible flow (suffix c), their geometries are related by (see [1, 2])

$$(\text{Camber})_c = \lambda(\text{Camber})_i \tag{2.6}$$

and

$$(\text{Maximum thickness})_c = \lambda(\text{Maximum thickness})_i \tag{2.7}$$

This rule is extended to include other geometries such as

(i) Chord *C*

$$(\text{Chord})_c = \lambda(\text{Chord})_i \tag{2.8}$$

(ii) Wing area *A*

$$(\text{Wing area})_c = \lambda(\text{Wing area})_i \tag{2.9}$$

(iii) Span *S*

$$(\text{Span})_c = \lambda(\text{Span})_i \tag{2.10}$$

2.3 Ratios of geometric characteristics for related aerofoils

For same aerofoil, let the ratio, $\frac{Maximum\ thickness}{Chord} = \beta$ (2.12)

and the ratio, $\frac{Camber}{Chord} = \gamma$ (2.12)

Also, by extension, let the ratio, $\frac{Wing\ area}{Spand} = \tau$ (2.13)

Then, for the related aerofoil, this ratio would become

$$\frac{Maximum\,thicknes}{Chord}s = \beta\left(1 - Ma^2\right)^{-\frac{1}{2}} = \beta\lambda \tag{2.14}$$

and

$$\frac{Camber}{Chord} = \gamma\left(1 - Ma^2\right)^{-\frac{1}{2}} = \gamma\lambda) \tag{2.15}$$

Similarly, $\frac{Wing\ area}{Spand} = \tau(1 - Ma)^{-\frac{1}{2}} = \tau\lambda$ (2.16)

2.4 Gothert's rules for related aerofoils

For such related aerofoils, similarity rules due to Gothert [3] are

$$(C_L)_c = \lambda^2 (C_L)_i \qquad (2.17)$$

for lift coefficient. For velocities, angles of incidence and pressures the rules are respectively the same as in

(2.2), (2.4) and (2.5). Thus, the expressions (2.1) – (2.17) constitute the model for simulating flow over aerofoils.

3.0 Numerical illustration

Consider, for example, a two-dimensional aerofoil having span of 10m, chord of 1.2m and mean chord of 2m. It has a ratio of maximum thickness to chord equal to 0.035, camber to chord ratio equal to 0.015 and wing area to span ratio equal to 2. It is tested in a low-speed wind tunnel at $100ms^{-1}$ and pressure 2000 Nm^{-2} and gives a lift coefficient of 0.35 at an angle of incidence of 3^0. We wish to determine (a) the velocity (*b*) the lift coefficient (c) the angle of incidence (d) the pressure of this aerofoil at $Ma = 0.5$. We also wish to determine (*e*) the velocity (*f*) the lift coefficient (*g*) the angle of incidence (*h*) the pressure (*i*) the geometric characteristics like maximum thickness, camber, chord, wing area, span and (*j*) the ratios like thickness/chord ratio, camber/chord ratio and wing area/span ratio for a related aerofoil at $Ma = 0.5$.

3.1 Solution

3.1.1 Same Aerofoil

Assume that the conditions at low-speed wind tunnel correspond to incompressible flow. From the problem $u_i = 100ms^{-1}$, $(C_L)_i = 0.35$, $\alpha_i = 3^0$, $p_i = 2000Nm^{-2}$, $Ma = 0.5$ giving $\lambda = 1.1547$ (see (2.1)). Substituting these data in the model expressions (2.2) – (2.5) gives respectively, $u_c = 115.47ms^{-1}$, $(C_L)_c = 0.4041$, $\alpha_c = 3.464^0$, $p_c = 2309.4Nm^{-2}$

3.1.2 Related aerofoil

From the problem [see (2.11) – (2.13)], $\beta = 0.035$, $\gamma = 0.015$, $\tau = 2$. Using the values of β, γ with chord = 1.2m respectively in (2.11), (2.12), and the value of τ with span = 10m in (2.13) we have $(\text{thickness})_i = 0.042$m, $(\text{camber})_i = 0.018$m, $(\text{wing area})_i = 20m^2$. Also, from the problem $(\text{chord})_i = 1.2$m and $(\text{span})_i = 10$m. Substituting the above in (2.6) – (2.10) we find

$(\text{camber})_c = 0.02078$m, $(\text{thickness})_c = 0.0485$m,
$(\text{chord})_c = 1.3856$m, $(\text{wing area})_c = 23.094m^2$
$(\text{span})_c = 11.547$m

Again, $\beta = 0.035$, $\gamma = 0.015$, $\tau = 2$.

Substituting these in the expressions (2.14) – (2.16) (using $\lambda = 1.1547$) we find respectively, for the related aerofoil, thickness/chord = 0.04041, camber/chord = 0.01732, wing area/span = 2.3094

Finally, using [3], its lift coefficient is obtained by substituting the appropriate data above in the model expression (2.17), while its velocity, angle of incidence and pressure are obtained respectively by substituting the appropriate data in (2.2), (2.4) and (2.5). Thus, we obtain

$C_L)_c = 0.4666$, $u_c = 115.47ms^{-1}$, $\alpha_c = 3.464^o$, $p_c = 2309.4\ Nm^{-2}$

3.3 Result

The new parameters (a) – (j) that are determined for same aerofoil are shown in Table 3.1 and can be compared with their counterparts for the related aerofoil in Table 3.2.

Table 3.1: Result for same aerofoil

	Same aerofoil	
Parameters	**Incompressible flow**	**Compressible flow**
Velocity	$100ms^{-1}$	$115.47ms^{-1}$
Lift coefficient	0.35	0.4041
Angle of incidence	3^0	3.464^0
Pressure	2000 Nm^{-2-}	$2309.4Nm^{-2}$
$(\text{Camber})_i$	0.018m	
$(\text{Thickness})_i$	0.042m	
$(\text{Chord})_i$	1.2m	
$(\text{Wing area})_i$	$20m^2$	
$(\text{Span})_i$	10m	

Thickness/chord ratio	0.035
Camber/chord ratio	0.015
Wing area/span ratio	2

Table 3.2: Result for related aerofoil

	Related aerofoil	
Parameters	**Incompressible flow**	**Compressible flow**
Velocity	$100ms^{-1}$	$115.47ms^{-1}$
Lift coefficient	0.35	0.4666
Angle of incidence	3^0	3.464^0
Pressure	$2000\ Nm^{-2-}$	$2309.4Nm^{-2}$
$(\text{Camber})_c$	0.02078m	
$(\text{Thickness})_c$	0.0485m	
$(\text{Chord})_c$	1.3856m	
$(\text{Wing area})_c$	$23.094m^2$	
$(\text{Span})_c$	11.547	
Thickness/chord ratio	0.04041	
Camber/chord ratio	0.01732	
Wing area/span ratio	2.3094	

4.0 Discussion and conclusion

Tables 4.1 and 4.2 show respectively the result of the analysis of the flow problem for same and related aerofoils. Comparison of the two Tables indicates that certain parameters, namely, velocity, angle of incidence and pressure for same aerofoil are the same as those for the related aerofoil in compressible flow. This is in agreement with the model (2.2), (2.4) and (2.5) respectively. On the other hand, the lift coefficient for the related aerofoil is numerically greater than its counterpart for same aerofoil in compressible flow. This also agrees with (2.3). Furthermore, it is observed from the two Tables that certain geometries like camber, maximum thickness, chord, wing area and span together with geometric ratios like thickness/chord ratio, camber/chord ratio and wing area/span ratio for the related aerofoil are greater numerically than those for same aerofoil. This is justified respectively by the expressions (2.6) – (2.10) and (2.14) – (2.16).

Finally, comparison of the parameters in compressible and incompressible flows shows that, generally, parameters in compressible flow are numerically greater than those in the incompressible flow in the two Tables. For instance, the high pressure in compressible flow in the two Tables implies a high lift force acting on the aircraft wing, and hence a higher aerodynamic performance in the compressible flow than in the incompressible flow.

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