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# Shear fields in a material weakened by cracks at the boundary of a hole

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***Abstract***

*A sheared elastic material containing a central hole and two cracks has been investigated. One of the cracks propagated from a far distance and terminated at the hole which acted as a crack breaker. The other one originated from the boundary of the hole opposite the one of infinite extent and moved a finite distance, b-a, where a is the radius of the hole. The fields close to the tip of the finite crack were derived in terms of the complete elliptic integral of the first kind. The stress intensity factor,* $K_{III}(b-a;T)$ *was derived in the standard form and is comparable with known stress intensity factor,* $K_{III}^{0}(a,T)$ *for a tunnel crack of width,* 2*a under remote shear stress. The dependence of* $K_{III}^{N}(b-a;T)=\dfrac{K_{III}(b-a;T)}{K_{III}^{0}(a;T)}$ *on the ratio* $\dfrac{a}{b}$ *was displayed in a graph.*

## 1.0 Introduction

This study concerns a homogeneous isotropic elastic material containing a central circular hole of radius, a. One of the cracks is of infinite extent and terminates at the boundary of the hole, which acts as a crack breaker while the other one originates from the side of the hole opposite the first one and has length $b - a$. The problem is formulated in terms of polar coordinates $r$, θ for the only non vanishing component of displacement, $W(r, \theta)$. The crack of infinite extent lies along the ray $r \geq a$, $\theta \pm \pi$ while the finite one lies on θ = 0, $a \leq r \leq b$. The boundary $r = a$, $0 < \theta < \pi$ is subjected to anti-plane shear, $-T$ while $r = a$, $-\pi < \theta < 0$ is subjected to opposite shear, $T$. The crack surfaces are stress (Figure 1.1). The central circular hole with finite cracks has been investigated by several authors see for example [1,2,3]. Investigation of anti-plane fields complements those of tensile and in-plane shear fields.

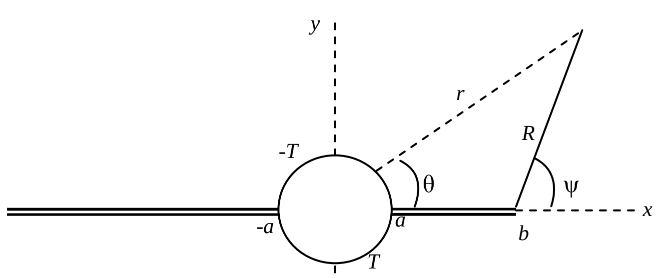



**Figure 1.1**: Geometry of the problem and load sites

## 2.0 Governing equations

The non-vanishing polar stresses are related to $W(r, \theta)$ through

$$\sigma_{\theta z}(r,\theta)=\frac{\mu}{r}\frac{\partial W}{\partial \theta}(r,\theta);\sigma_{rz}(r,\theta)=\mu\frac{\partial W}{\partial r}(r,\theta) \quad (2.1)$$

$W(r, \theta)$. is sought for in the following boundary value problem:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)W(r,\theta)=0,\ -\pi\le\theta\le\pi,\ r\ge a \quad (2.2)$$

$$\frac{\partial W}{\partial r}(a,\theta)=-\frac{T}{\mu},\ \ 0<\theta<\pi \quad (2.3a)$$

$$\frac{\partial W}{\partial r}(a,\theta)=\frac{T}{\mu},\ \ -\pi<\theta<0 \quad (2.3b)$$

$$\frac{\partial W}{\partial \theta}(r,\theta)=0,\ \ \theta=0,\ a\le r\le b\ \ and\ \ \theta=\pm\pi,\ r>a \quad (2.4)$$

The boundary and the problem are made more mathematically tractable by their transformation using the conformal mapping function

$$\xi(z)=\frac{1}{2}\left(\frac{z}{a}+\frac{a}{z}-\frac{b}{a}-\frac{a}{b}\right);\ z=re^{i\theta} \quad (2.5)$$

Let $\xi(z)=u(r,\theta)+iv(r,\theta)=\rho e^{i\phi}$, then $u(r,\theta)=\frac{1}{2}\left[\left(\frac{r}{a}+\frac{a}{r}\right)\cos\theta-\frac{b}{a}-\frac{a}{b}\right]=\rho\cos\phi$

$v(r,\theta)=\frac{1}{2}\left(\frac{r}{a}-\frac{a}{r}\right)\sin\theta=\rho\sin\phi$, $\rho(r,\theta)=\left[u^2(r,\theta)+v^2(r,\theta)\right]^{\frac{1}{2}}$ $and$ $\tan\phi=\frac{v(r,\theta)}{u(r,\theta)}$.

From these coordinates we deduce $\frac{\partial p}{\partial \theta}(r,\pm\pi)=0;\ \frac{\partial p}{\partial \theta}(r,0)=0;\ \frac{\partial \rho}{\partial r}(a,\theta)=0$

$$\frac{\partial \phi}{\partial \theta}(r,\pm\pi)\neq 0;\ \frac{\partial \phi}{\partial r}(a,\theta)=-\frac{\sin\theta}{a(\in-\cos\theta)} \quad (2.6)$$

where $\in=\frac{1}{2}\left(\frac{b}{a}+\frac{a}{b}\right)$. Similarly $\rho(a,\theta)=\in-\cos\theta$, implies $-1\le\in-\rho\le1$, hence

$$\frac{\partial \phi}{\partial r}(a,\theta)=\pm\frac{\sqrt{1-(\rho-\in)^2}}{a\rho}\quad \rho\le 1+\in,-\pi\le\theta\le\pi$$

$W(r,\theta)\equiv W(\rho,\phi)$ and (2.6) imply

$$\frac{\partial W}{\partial r}(a,\theta)=\frac{\partial W}{\partial \phi}(\rho,\pm\pi)\frac{\partial \phi}{\partial r}(a,\theta)\ \in-1<\rho<\in+1,-\pi\le\theta\le\pi$$

Therefore, we seek $W(\rho,\phi)$ in the problem (see Figure 2.2)

$$\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right)W(\rho,\phi)=0,\ -\pi\le\phi\le\pi,\ \rho\ge0 \quad (2.7a)$$

$$\frac{\partial W}{\partial \phi}(\rho,\pm\pi)=\frac{aT}{\mu}\frac{\rho}{\sqrt{1-(\rho-\in)^2}},\ \in-1<\rho<\in+1 \quad (2.7b)$$

$$= 0 \text{ otherwise} \tag{2.7c}$$

The behaviour of the stresses are

$$\sigma_{\rho z}(\rho,\phi) = \sigma_{\phi z}(\rho,\phi) = \begin{cases} 0\left(\rho^{\frac{-1}{2}}\right) & as \quad \rho \to 0 \\ 0\left(\rho^{-\frac{3}{2}}\right) & as \quad \rho \to \infty \end{cases}$$

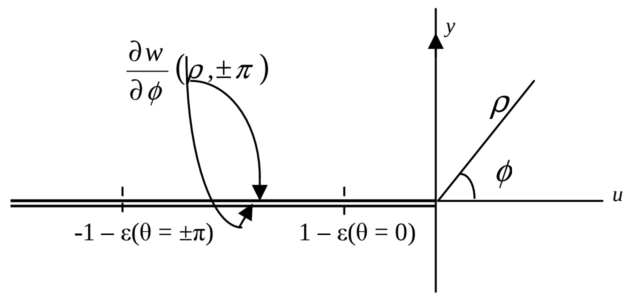



**Figure 2.1**: Boundary conditions due to the transformation

## 3.0 Solution of the transformed problem

Mellin transformation of (2.7) yields

$$\left(\frac{d^2}{d\phi^2} + s^2\right)\overline{W}(s,\phi) = 0 \quad -\frac{1}{2} < \operatorname{Re} s < \frac{1}{2} \tag{3.1}$$

$$\frac{d\overline{W}}{d\phi}(s,\pm\pi) = \frac{aT}{\mu} g(a,b;s) \tag{3.2}$$

where the Mellin transform of $W(\rho, \theta)$ is defined by

$$\overline{W}(s,\phi) = \int_0^\infty W(\rho,\phi)\rho^{s-1} d\rho \quad -\frac{1}{2} < \operatorname{Re} s < \frac{1}{2} \tag{3.3}$$

and

$$g(a,b;s) = \int_{\in-1}^{\in+1} \frac{\rho^s}{\sqrt{1-(\rho-\in)^2}} d\rho \tag{3.4}$$

To analyze the singularities of $g(a,b;s)$ we write the integrand as

$$\rho^s\left(1-(\rho-\in)^2\right)^{-\frac{1}{2}} = \rho^s\{1-(\rho-\in)\}^{-\frac{1}{2}}\{1+\rho-\in\}^{-\frac{1}{2}}$$

$$= \rho^s(1+\in-\rho)^{-\frac{1}{2}}(1-\in+\rho)^{-\frac{1}{2}} = \rho^{s-\frac{1}{2}}(1+\in)^{-\frac{1}{2}}(1-\alpha\rho)^{-\frac{1}{2}}\left(1-\beta\rho^{-1}\right)^{-\frac{1}{2}}$$

where $\alpha = \frac{1}{1+\in}$ and $\beta = \in -1$. The integral may then be evaluated by use of the series expression

$$(1-t)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} b_k t^k, \ |t| < 1, \ b_k = \frac{(2k)!}{2^{2k}(k!)^2}$$

with respect to which the integrand is then written as the double series

$$\rho^s\left[1-(\rho-\in)^2\right]^{-\frac{1}{2}}=(\in+1)^{-\frac{1}{2}}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\alpha^k\beta^m b_k b_m \rho^{s+k-m-\frac{1}{2}}$$

Hence $$g(a,b;s)=(\in+1)^{-\frac{1}{2}}\left\{\sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty}\alpha^k\beta^m b_k b_m\left(\frac{(\in+1)^{s+k-m+\frac{1}{2}}-(\in-1)^{s+k-m+\frac{1}{2}}}{s+k-m+\frac{1}{2}}\right)\right.$$

$$+\sum_{\substack{m=0\\k=m}}^{\infty}\sum_{k=0}^{\infty}\alpha^k\beta^m b_k b_m\left(\frac{(\in+1)^{s+k-m+\frac{1}{2}}-(\in-1)^{s+k-m+\frac{1}{2}}}{s+k-m+\frac{1}{2}}\right)$$

$$\left.+\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}\alpha^k\beta^m b_k b_m\left(\frac{(\in+1)^{s+k-m+\frac{1}{2}}-(\in-1)^{s+k-m+\frac{1}{2}}}{s+k-m+\frac{1}{2}}\right)\right\} \tag{3.5}$$

Equation (3.5) is obtained from

$$g(a,b;s)=(\in+1)^{-\frac{1}{2}}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\alpha^k\beta^m b_k b_m\left(\frac{(\in+1)^{s+k-m+\frac{1}{2}}-(\in-1)^{s+k-m+\frac{1}{2}}}{s+k-m+\frac{1}{2}}\right) \tag{3.6}$$

to separate singularities because they may not occur, in the cases $k>m, k=m$ and $k<m$, in the same half planes, *Res* > 0 and *Res* < 0. Let

$$M(t;s)=\sum_{m=0}^{\infty}\sum_{k=m+1}^{\infty}\alpha^k\frac{\beta^m b_k b_m}{s+k-m+\frac{1}{2}}t^{k-m+\frac{1}{2}}+\sum_{\substack{m=0\\k=m}}^{\infty}\sum_{k=0}^{\infty}\frac{\alpha^k\beta^m b_k b_m t^{\frac{1}{2}}}{s+\frac{1}{2}}+\sum_{k=0}^{\infty}\sum_{m=k+1}^{\infty}\frac{\alpha^K\beta^M b_K b_m t^{k-m+\frac{1}{2}}}{s+k-m+\frac{1}{2}}$$

Then (3.5) becomes $g(a,b;s)=(\in+1)^{-1/2}\left\{M(\in+1;s)(\in+1)^s-M(\in-1;s)(\in-1)^s\right\}$ (3.7)

If (3.2) is applied to the solution of (3.1) given as $\overline{W}(s,\phi)=A(s)\sin\phi s+B(s)\cos\phi s$ (3.8)

it turns out that $B(s)=0$ *and* $A(s)=\dfrac{aT\,g(a,b;s)}{\mu s\cos\pi s}$. Hence

$$\overline{W}(s,\phi)=\frac{aT}{\mu}g(a,b;s)\frac{\sin\phi s}{s\cos\pi s} \tag{3.9}$$

The inversion formula for the Mellin transform and (3.9) yield

$$W(\rho,\phi)=\frac{aT}{\mu}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}g(a,b;s)\rho^{-s}\frac{\sin\phi s}{s\cos\pi s}ds,\ -\frac{1}{2}<c<\frac{1}{2} \tag{3.10}$$

Using (3.11) the integral in (3.10) is evaluated for ε - 1 < $\rho$ < $\varepsilon$ + 1 by residue technique with choice of contours regulated by Jordan's lemma. It is note worthy that the singularities involved do not occur simultaneously when $\rho$ < $\varepsilon$ + 1 and ρ > $\varepsilon$ -1. For $\rho$ < $\varepsilon$ + 1, Jordan's lemma suggests closure of contour in the left half plane *Res* < 0 where $M(\varepsilon + 1; s)$ has double poles at $s = -(k - m) - ½$, $k > m$ and at $s = ½$. For $\rho > \varepsilon - 1$ the contour is closed in the right half plane *Res* > 0 where $M(\varepsilon - 1; s)$ has double poles at $s = m - k - ½$, $k < m$. The integrals to be evaluated are:

$$I_{\in+1}(\rho,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}M(\in+1;s)\left(\frac{\rho}{\in+1}\right)^{-s}\frac{\sin\phi s}{s\cos\pi s}ds,\ \rho<\in+1$$

$$I_{\in-1}(\rho,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}-M(\in-1;s)\left(\frac{\rho}{\in-1}\right)^{-s}\frac{\sin\phi s}{s\cos\pi s}ds\quad \rho>\in-1$$

The solution for $\in-1<\rho<\in+1$ is then written as $W(\rho,\phi)=\dfrac{aT}{\mu(\in+1)^{\frac{1}{2}}}\{I_{\in+1}(\rho,\phi)+I_{\in-1}(\rho,\phi)\}$

where $$I_{\in+1}(\rho,\phi)=\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\alpha^{m+n-1}\beta^{m}b_{m+n-1}b_{m}\left\{\begin{matrix}\left[\dfrac{2}{2n-1}-\ln\left(\dfrac{\rho}{\in+1}\right)\right]\sin\left(n-\dfrac{1}{2}\right)\phi\\ -\phi\cos\left(n-\dfrac{1}{2}\right)\phi\}\rho^{n-\frac{1}{2}}\end{matrix}\right.$$

and $$I_{\in-1}(\rho,\phi)=\frac{2}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\alpha^{m}\beta^{m+n}b_{m}b_{m+n}\left\{\begin{matrix}\left[\dfrac{2}{2n-1}+\ln\left(\dfrac{\rho}{\in-1}\right)\right]\sin\left(n-\frac{1}{2}\right)\phi\\ -\phi\cos\left(n-\frac{1}{2}\right)\phi\end{matrix}\right\}\rho^{\frac{1}{2}-n}$$

## 4.0 Crack tip neigbourhood fields

The solution for $0<\rho<1$ is derived by employing (3.6) and noting that, in this case $g(a,b's)$ is an analytic function with removable singularities. The integral in (3.10) is therefore evaluated by referring to the simple poles contributed by $\cos\pi s$ in the left half plane *Res* < 0. The result is:

$$W(\rho,\phi)=\frac{2aT}{\mu\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}g\left(a,b;\frac{1}{2}-n\right)\rho^{n-\frac{1}{2}}\sin\left(n-\tfrac{1}{2}\right)\phi,\ \rho<1 \qquad (4.1$$

As the crack tip is approached (4.1) yields

$$W(\rho,\phi)=\frac{2aT}{\mu\pi}g\left(a,b;\frac{-1}{2}\right)\rho^{\frac{1}{2}}\sin\frac{\phi}{2}\quad as\quad \rho\to 0 \qquad (4.2)$$

It follows from (3.4) that $g\left(a,b;-\dfrac{1}{2}\right)=\displaystyle\int_{\in-1}^{\in+1}\frac{d\rho}{\sqrt{\rho(\rho+1-\in)(1+\in-\rho)}}$. By entry 3.1316 [4] the last expression is

$$g\left(a,b;-\frac{1}{2}\right)=\frac{2}{(\in+1)^{\frac{1}{2}}}F\left(\frac{\pi}{2},\sqrt{\frac{2}{\in+1}}\right) \qquad (4.3)$$

where $F\left(\dfrac{\pi}{2},p\right)$ is the complete elliptic integral of the first kind whose values are tabulated, for example in [5]. Introducing local polar coordinates $(R,\psi)$ at the crack tip, (Figure 2.1), we get $r\cos\theta = b + R\cos\psi$ and $r\sin\theta = R\sin\psi$ which applied to (2.5) yields $\rho e^{i\phi}=\dfrac{1}{a}\left(1-\dfrac{a^2}{b^2}\right)\mathrm{Re}^{i\psi}+0\left[\left(\dfrac{R}{a}\right)^2\right]\quad as\quad R\to 0$

That is $$\rho^{\frac{1}{2}}\sin\frac{\phi}{2}=\left[\frac{1}{a}\left(1-\frac{a^2}{b^2}\right)\right]^{\frac{1}{2}}R^{\frac{1}{2}}\sin\frac{\psi}{2}\quad as\quad R\to 0$$

In view of these derivations and (4.2), the standard from of the displacement is [6]

$$W(R,\psi)=\left(\frac{2a}{\pi}\right)^{\frac{1}{2}}\left(1-\frac{a^2}{b^2}\right)^{\frac{1}{2}}\frac{T}{\mu}g\left(a,b;\frac{-1}{2}\right)\left(\frac{2R}{\pi}\right)^{\frac{1}{2}}\sin\frac{\psi}{2}\quad as\quad R\to 0 \qquad (4.4))$$

The stress intensity factor is then given by $K_{III}(b-a;T)=(b-a)^{\frac{1}{2}}T\left[\frac{2}{\pi}\frac{a}{b}\left(1+\frac{a}{b}\right)\right]^{\frac{1}{2}}g\left(a,b;\frac{-1}{2}\right)$ (4.5)

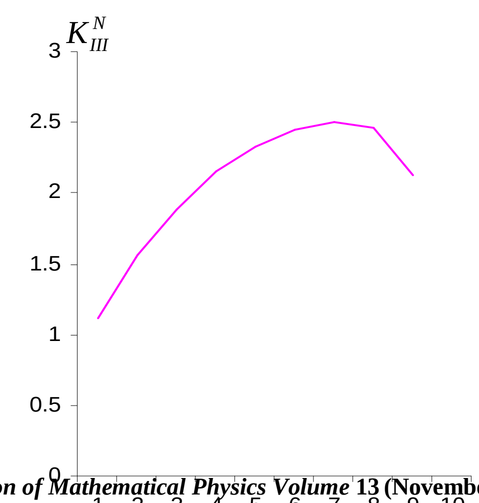
The energy release rate is $G=\frac{1}{2\mu}K_{III}^{2}(b-a;T)$

## 5.0 Conclusion

Noting that $\frac{\in+1}{2}=\frac{(a+b)^2}{4ab}$ and because the harmonic mean of positive numbers is less than their arithmetic mean, we find that $\left(\frac{2}{\in+1}\right)^{\frac{1}{2}}=\frac{\sqrt{ab}}{\left(\frac{a+b}{2}\right)}<1$, implies that $F\left(\frac{\pi}{2},\frac{2\sqrt{ab}}{a+b}\right)$ is valid. The displacement field for the geometry under consideration has been derived in (4.4) in terms of the complete elliptic integral of the first kind for b>a. The stress intensity factor is

$$K_{III}(b-a;T)=\left(\frac{b-a}{b+a}\right)^{\frac{1}{2}}\left(\frac{a}{b}\right)^{\frac{1}{2}}4\left(\frac{a}{\pi}\right)^{\frac{1}{2}}TF\left(\frac{\pi}{2},\frac{2\sqrt{ab}}{a+b}\right),\ b>a$$

Define $K_{III}^{N}(b-a;T)=\frac{K_{III}(b-a,T)}{K_{III}^{0}(a;T)}$, where $K_{III}^{0}(a;T)=(\pi a)^{\frac{1}{2}}T$ is the known stress intensity factor for a tunnel crack of width 2*a* in an infinite body with anti plane shear load applied parallel to the tunnel [1]. Then $K_{III}(a-b;T)$ can be compared with known stress intensity results. The dependence of $K_{III}^{N}$ on the ratio $\frac{a}{b}$ is showed in Figure 5.1.



$\frac{a}{b}$

**Figure** Dependence Of $K_{III}^{N}$ on $\frac{a}{b}$

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