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Linear stability analysis of a radiating fluid layer with temperature-dependent viscosity heated from below

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Abstract

Experiments with fluid whose viscosity depends on temperature are used to study the effect of viscosity variations. At large viscosity, Rayleigh number can either be conductive or convective depending on whether the Rayleigh number is high or low or decreased from a pre-existing convective state. At high Rayleigh numbers and for the entire range of viscosity variation, the stability occurs at $Ra_{cri} = 407.70$. ***From the viscous fluid when the Rayleigh is defined in terms of wave number corresponding to a temperature equal to the average of the boundary temperatures. The relationship between critical Rayleigh number and the temperature appears not to depend on the Rayleigh number.***

1.0 Introduction

Although the Earth's mantle, which is a spherical shell between the lithospheric layer and the core is largely a solid and behaves elastically on short time scales, it acts like an extremely viscous fluid on longer time scales and its convection is driven thermally and compositionally. The Earth's mantle viscosity could be dependent on temperature that could impact greatly on the style of convection, which with increasing viscosity contract is larger than 10^4 - 10^5. Min Chan et al (2004) considered theoretically an initially quiescent, fluid-saturated , horizontal porous layer heated from below with constant heat flux, use darig's law as a model to the fluid motion and linear stability theory, predict the onset of buoyancy-driven convective motion.[2] using propagation theory on the onset of buoyancy-driven convection in an initially isothermal quiescent fluid layer confined between the two infinite horizontal plates, predicts that dimensionless critical time t decreases with increasing prenatal number for a given Rayleigh number. [3], investigated how variable viscosity affects the onset of instability in the Rayleigh-Benard convection using the so called combine method. [4], said, the problem of heat transfer by convection in a horizontal layer of fluid across which the imposed temperature gradient gives rise to an unstable density gradient is classical in the theory of fluid mechanics. [5], experiments with fluids whose viscosity depends strongly on temperature are used to study the effect of viscous variations in the range 10^4 - 10^5on the heat transfer and horizontally averaged temperature of a convecting layer between horizontal isothermal boundaries. [6], treated the problem for a gray fluid with black rigid boundaries and arbitrary optical thickness using rigorous integral formulation of radiative transfer, but he neglected both thermal conduction and the radiation effect on the static temperature. [7] were the first to introduce the effects of fluid nongrayness and boundary emissivities. They used the approach of [8] , combining planck and Rossel and mean absorption coefficients, to account for fluid nongrayness.[9], in their work, investigated theoretically the radiation on the Rayleigh-Benard instability for real gases such as NH_3, H_20 or $C0_2$, they develop a procedure that enables quantitative predictions of critical Rayleigh numbers and that accounts for the complex structure of molecular absorption spectra and the full integro- differential nature of the radiative transfer equation. The main aim of this paper is to examine the linear analysis of the problem of a radiating fluid layer heated from below with temperature-dependent viscosity and application to the Earth's mantle; since the rheology of mantle material is likely to be temperature dependent, critical Rayleigh numbers considered and the temperature appeared not to depend on the Rayleigh number.

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2.0 Basic formulation

We consider the problem of linear stability analysis of an incompressible viscous radiating fluid with temperature-dependent viscosity in a horizontal channel with radiating heated flux. The system used in this paper is a radiating fluid with an initial temperature. The horizontal layer of fluid depth is d.

Under Boussinesq approximation, the governing equations for the flow including radiative heat transfer are

$$\nabla . V = 0 \tag{2.1}$$

$$\rho \frac{DV^1}{Dt^1} = \nabla p^1 + \rho g \beta \left(T^1 - T_\infty\right) \overset{o}{e}_z + \mu \nabla^2 V \tag{2.2}$$

$$\rho c_p \frac{DT^1}{Dt^1} = \kappa \nabla^2 T^1 - \nabla . q_r \tag{2.3}$$

where $V^1, T, p, \rho, \beta, g, \beta, \mu, and, q_r$ are respectively the velocity vector, temperature, pressure, density, thermal expansion coefficient, gravitational acceleration constant, viscosity and radiation heat. The subscript q_r represents basic state. The surface temperature $\tilde{\hat{e}}_z$ at the vertical distance z = 0 increases with the time during the conduction state . The fluid temperature is prescribed at the boundaries while velocity boundary conditions are determined either from the no slip condition for a rigid surface or for a free surface. The basic state of the system by the static solution $V = 0$ of the system (2.1) – (2.3) to which correspond the static temperature T, and the radiative vertical heat flux q_r. In other to study the stability of the static state, and thus, to determine the onset condition of Rayleigh–Benard instability, we consider the perturbed state defined by the layers

$$V = V^1 \tag{2.4a}$$

$$T^1 = T_{0(z)} + T_1 \tag{2.4b}$$

$$P^1 = P_0 + P_1 \tag{2.4c}$$

where $T << T_0$, $P_0 <<< P_1$.. The temperature disturbance induces a disturbance q_r in the radiative flux which becomes

$$\nabla . q_r = \alpha^2 (T^1 - T_\infty) \tag{2.4d}$$

We follow the classical procedure of linear mode analysis of [10]. The decomposition given in (2.4) is introduced in (2.1) – (2.3) and the resulting equations are linearized.

The linear system is then reduced to a set of two scalar equations by taking the double curl of the momentum equation and keeping only the vertical component w of the velocity, we have two equations

$$\left(\frac{\partial}{\partial t^1} - \frac{\mu}{\rho}\nabla^2\right)\nabla^2 w_1 = g\beta \nabla_n^2 T_1 - \frac{1}{\rho}\left(\frac{\partial^2 \mu}{\partial x^{12}} + \frac{\partial^2 \mu}{\partial y^{12}} + \frac{\partial^2 \mu}{\partial z^{12}}\right) - \frac{1}{\rho}\frac{\partial^2 \mu}{\partial z^{12}}\left(\frac{\partial^2 w_1}{\partial x^{12}} + \frac{\partial^2 w_1}{\partial y^{12}} + \frac{\partial^2 w_1}{\partial z^{12}}\right) \tag{2.5}$$

where $\nabla_n^2 = \frac{\partial^2}{\partial x^{12}} + \frac{\partial^2}{\partial y^{12}}$. And the other

$$\frac{\partial T_1}{\partial t^1} + w_1 \frac{\partial T_0}{\partial z^1} = \frac{\partial^2 T_1}{\partial z^{12}} - \frac{\alpha^2 T_1}{\kappa} \tag{2.6}$$

Using the non-dimensional parameters

$$t^1 = \frac{d^2 t}{\alpha}, \alpha = \frac{\mu}{\rho}, \nabla^{12} = \nabla^2, w_1 = w\alpha, T_1 = \frac{qw\alpha d^2\theta}{\lambda}, \mu = \mu_0 \mu x^1 = dx, y^1 = dy, z^1 = dz \tag{2.7}$$

(2.5) after simplifying becomes

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w = \frac{Ra}{\Pr}\nabla_n^2 \theta - \nabla^2 \mu - \nabla^2 w \tag{2.8}$$

Also, using the non-dimensional parameters

$$t=\frac{\alpha}{d^2}t^1,(u,v,w)=\frac{1}{\alpha}(u,v,w)\theta=\frac{\lambda T_1}{qwd^2\alpha},\theta_0=\frac{\lambda T_0}{d^2\alpha qw},\lambda=\frac{\kappa}{ec_p},(x,y,z)=\left(\frac{x^1}{d},\frac{y^1}{d},\frac{z^1}{d}\right) \quad (2.9)$$

(2.6) after simplifying becomes
$$\Pr\left(\frac{\partial\theta}{\partial t}+\alpha_s w\right)=\left(\frac{\partial^2}{\partial z^2}-B^2\right)\theta \quad (2.10)$$

where $\alpha_s=\frac{\partial\theta_0}{\partial z}$ with the boundary conditions at z = ± 1

$$\left.\begin{array}{l}\theta=w=0\\ \frac{\partial w}{\partial z}=0, \text{ on a rigid surface}\\ \frac{\partial^2 w}{\partial z^2}=0, \text{ on a free surface}\end{array}\right\} \quad (2.11)$$

Following Chandrasekhar (1961), We study the stability of normal mode disturbances, which are chosen in our case to be two dimensional periodic waves in the horizontal plane (*x*, *y*). Each normal mode is defined by a non dimensional wave number *K*, and the corresponding disturbances have temporal and spatial horizontal dependence of the form $\exp[i(k_x x+k_y y)+nt]$, where $k^2=k_x^2+k_y^2$. We seek for solutions of the form

$$w=W_{(z)}\exp[i(k_x x+k_y y)+nt] \quad (2.12)$$

$$\theta=\Theta_{(Z)}\exp\geq[i(k_x x+k_y y)+nt] \quad (2.13)$$

Substitute (2.12) and (2.13) into (2.8) and (2.10), differentiate with respect to t, x, y, z and simplify, we get

$$\left[\left(\frac{d^2}{dz^2}-K^2\right)\left(\frac{d^2}{dz^2}-(n+K^2)+1\right)\right]W=K^2\left(\frac{Ra}{\Pr}\right)\Theta \quad (2.14)$$

and
$$\left(\frac{d^2}{dz^2}-B^2-n\Pr\right)\Theta=\Pr\alpha_s W \quad (2.15)$$

3.0 Stability analysis

In order to determine the stability of system (2.14) and (2.15), we must study the stability of all possible disturbances for all the wave numbers. The stability criterion can be found by determining the Rayleigh number.

Let $D^2=\frac{d^2}{dz^2}, n=0,$ then (2.14) and (2.15) becomes

$$[(D^2-K^2)(D^2-K^2+1)]W=K^2\left(\frac{Ra}{\Pr}\right)\Theta \quad (3.1)$$

and
$$(D^2-B^2)\Theta=\Pr\beta\alpha_s W \quad (3.2)$$

With boundary conditions $\Theta=0, W=0,$ for z = 0 to 1 (3.3)

$D^2W=0,$ on a free surface (3.4)

Eliminating Θ from equations (3.1) and (3.2) we have

$$[(D^2-B^2)(D^2-K^2)(D^2-K^2+1)]W=[K^2Ra\alpha_s]W \quad (3.5)$$

Let $W = A\,Sin\pi z,$ m = 1, 2,… (3.6) where A is a constant, m is an integer. Substitute (3.6) into (3.5), simplify and take the second derivative with respect to z will result in

$$Ra = \frac{\left(m^2\pi^2 + B^2\right)\left(m^2\pi^2 + K^2\right)\left(m^2\pi^2 + K^2 + 1\right)}{K^2\alpha_s} \tag{3.7}$$

For a given K^2, the lowest value of Ra occurs when m = 1, then

$$Ra = \frac{\left(\pi^2 + B^2\right)\left(\pi^2 + K^2\right)\left(\pi^2 + K^2 + 1\right)}{K^2\alpha_s} \tag{3.8}$$

But $\alpha_s = \frac{\partial\theta_0}{\partial z}$. The static temperature θ_0 and viscous heat flux B^2 fluid are related by the energy balance

$$\frac{d^2\theta_0}{dz^2} - B^2\theta_0 = 0 \tag{3.9}$$

Subject to $\theta_0 = \pm 1$ at $z = \mu 1$ (3.10)

The solution of (3.9) subject to conditions (3.10) is given by

$$\theta_0 = \frac{\frac{1}{2}Sinh(Bz)}{Sinh\left(\frac{B}{2}\right)} \tag{3.11}$$

$\frac{\partial\theta_0}{\partial z} = \frac{\frac{B}{2}Cosh(Bz)}{Sinh\left(\frac{B}{2}\right)}$. Therefore (3.8) becomes

$$Ra = \frac{\left(\pi^2 + B^2\right)\left(\pi^2 + K^2\right)\left(\pi^2 + K^2 + 1\right)}{K^2}\left[\frac{Sinh\left(\frac{B}{2}\right)}{\frac{B}{2}Cosh(Bz)}\right] \tag{3.12}$$

4.0 Results

From the analysis of stationary convection in the presence of radiation with temperature dependent, it is observed that critical Rayleigh number is minimum when the denominator is maximum. This value occurs at z = 0, that is, at the centre of the horizontal channel. For small values of the temperature parameter (i.e B = 0), our result reduces the onset criterion in terms of Rayleigh number determined by Ra_{cri} = 407.70 with wave number K = 3.2 for which the stability occurs in Figure 1. This is plus one above the results of [10]. Furthermore, little increase of the values of the temperature parameter (i.e B = 0.2, 0.5, 1.0), our results show that the stability occurs at K = 3.2, while the Ra_{cri} changes as shown in table 1, Fig.2, Fig.3 and Fig.4, which is plus one above the results of [10]

Table 4.1.

K	B = 0.1	B = 0.2	B = 0.5	B = 1
1	1268.83	1255.46	1169.94	947.15
1.1	1087.79	1076.33	1003.01	811.99

1.2	950.78	940.77	876.68	710.22
1.3	844.86	835.96	779.01	631.1
1.4	761.52	753.5	702.17	568.83
1.5	694.99	687.67	640.83	519.13
1.6	641.26	634.5	591.28	478.99
1.7	597.45	591.15	550.88	446.25
1.8	561.45	555.54	517.69	419.37
1.9	531.72	526.11	490.27	397.15

K	$B = 0.1$	$B = 0.2$	$B = 0.5$	$B = 1$
2	507.06	501.71	467.54	378.73
2.1	486.57	481.44	448.64	363.42
2.2	469.54	464.59	432.95	350.73
2.3	455.42	450.63	419.93	340.16
2.4	443.77	439.1	409.19	331.45
2.5	434.23	429.66	400.39	324.32
2.6	426.51	422.02	393.27	318.55
2.7	420.38	415.95	387.61	313.97
2.8	415.63	411.25	383.24	310.42
2.9	412.12	407.78	380	307.79
3	409.59	405.38	377.77	305.98
3.1	408.91	403.95	376.44	304.9
3.2	407.7	403.4	375.92	304.48
3.3	407.94	403.64	376.14	304.66
3.4	408.91	404.6	377.04	305.25
3.5	410.56	406.23	378.56	306.47
3.6	412.82	408.47	380.65	308.16
3.7	415.66	411.28	383.26	310.28
3.8	419.03	414.62	386.38	312.8
3.9	422.91	418.45	389.95	315.69
4	427.25	422.75	393.95	318.94
4.1	432.04	427.49	398.37	322.51
4.2	437.25	432.64	403.17	326.4
4.3	442.86	438.19	408.34	330.59
4.4	448.45	444.12	413.87	335.06
4.5	455.21	450.41	419.73	339.8
4.6	461.92	457.05	425.91	344.81
4.7	468.96	464.02	432.41	350.07
4.8	476.33	471.32	439.21	355.58
4.9	484.03	478.93	446.3	361.32
5	492.02	486.84	453.68	367.29

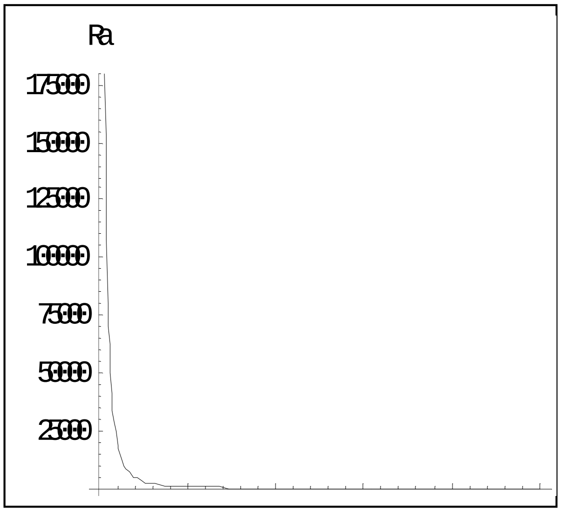

Figure 4.1: Graph of *Ra* against *K* at $B = 0.1$

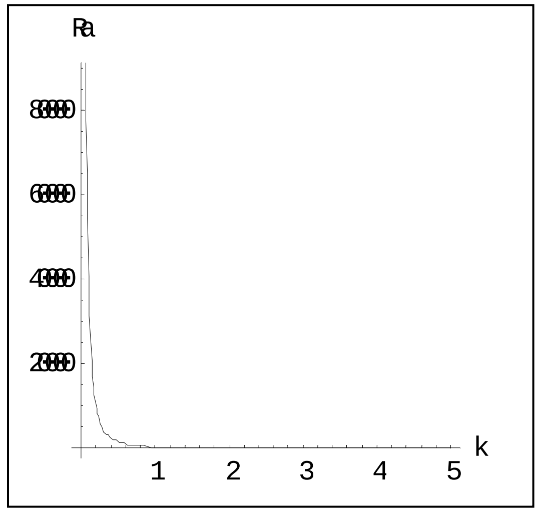

Figure 4.2: Graph of *Ra* against *K* at $B = 0.2$

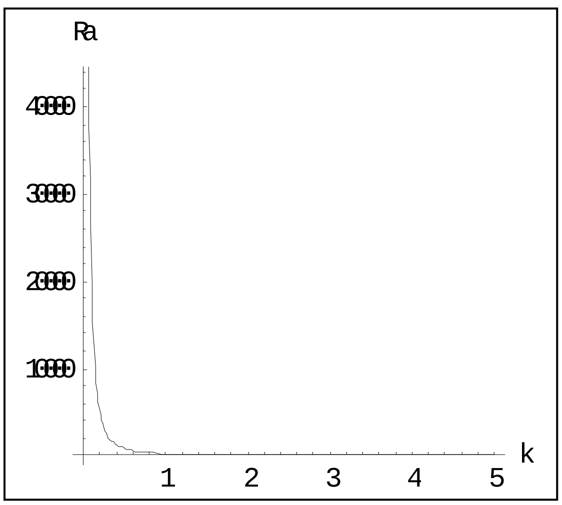

Figure 4 3: Graph of *Ra* against *K* at $B = 0.5$

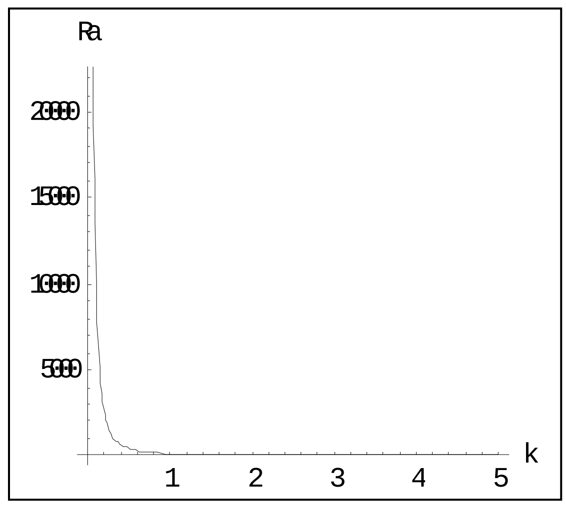

Figure 4.4: Graph of *Ra* against *K* at $B = 1.0$

5.0 Conclusion

We have presented a fundamental linear stability analysis for viscous fluid including the effects of temperature – dependent. The fluid flow considered using the Cartesian coordinate system from our presentation extends to infinity in the z direction and is two dimensional hence they are difficult to realize in application but being fundamental it forms the basis and are often used as good approximations. However, from table 1, it is observed that increase in temperature parameter, delays the onset of instability. Therefore, we conclude that a small variation in the temperature parameter stabilizes the radiating fluid layer heated from below.

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