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Acoustic wave in viscous fluid

Eghuanoye Ikata
Physics Department
College of Education, Port Harcourt, Nigeria.

Abstract

This discussion is concerned with an acoustic wave in a viscous Newtonian fluid and examines the wave equation in this case. The governing partial differential equations for the wave are derived using the Navier-Stokes momentum equation, the equation of continuity (assuming an adiabatic equation of state) and the acoustic wave condition. One advantage of this formulation is that the wave equation is expressed in a form suitable for finite difference time domain discretization. The formulation shows that, in general, acoustic wave propagation in a viscous fluid is associated with both shearing (tangential) and non-shearing (normal) viscous forces, which account for dissipation of wave energy in the medium. In a liquid or for a plane acoustic wave the non-shearing (normal) viscous force is the dominant contributor to wave energy dissipation.

Keywords: acoustic wave; viscous loss.

1.0 Introduction

An acoustic wave is a small variation in medium density or pressure which propagates in a compressible medium. In a medium that exhibits little restraint to deformation, such as a fluid, the restoring force responsible for acoustic wave propagation in the medium is simply due to a change in pressure. An acoustic wave is a common example of a longitudinal wave, or a scalar field as it can be completely represented using a single function. Using a field approach to describe the wave, the relevant field functions are the density $\rho(\mathbf{r},t)$, particle velocity $\mathbf{v}(\mathbf{r},t)$ and pressure $p(\mathbf{r},t)$, where $\mathbf{r}$ is a position vector and t is time. We use the word 'field' in the ordinary mathematical sense, which implies a function of space (and time), and not in the extended physical sense which implies a force field. The field functions $\rho(\mathbf{r},t)$, $\mathbf{v}(\mathbf{r},t)$ and $p(\mathbf{r},t)$ which describe an acoustic wave are related using the equation of continuity, the equation of motion and the equation of state.

Mathematically, wave phenomenon is represented by a wave equation. The acoustic wave equation is derived using the equation of continuity and the equation of motion. Usually, the acoustic wave equation is a second-order partial differential equation for any single function chosen to represent the scalar field. Acoustic wave propagation in a lossy medium leads to the dissipation of acoustic energy in the medium. When energy loss is involved, the loss mechanism may be introduced in either the equation of continuity or the equation of motion. Thus, in general, an acoustic wave equation in a lossy medium can be derived using either a lossy equation of motion and a lossless equation of continuity or a lossless equation of motion and a lossy equation of continuity. Earlier approaches made use of a lossless equation of motion and a lossy equation of continuity, obtained by using Stokes approximation to the equation of state [1]. Often, acoustic energy loss in a medium is associated with various mechanisms like viscosity, thermal conduction and molecular exchanges.

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Here we represent the acoustic wave equation in a viscous fluid using a lossless equation of continuity and a lossy equation of motion. Actually, the notion of a 'lossless' equation of continuity (as an expression of conservation of mass) is a generalization which is (often) true in wave phenomenon, since the concept of wave propagation excludes sources, sinks and mass transport (especially, in small amplitude waves). Such a formulation gives equations which may be discretized by the finite difference time domain method [2].

2.0 Equation of motion

Ignoring the temperature dependence of the viscosities in a Newtonian fluid, the general equation of motion is the Navier-Stokes (momentum) equation [3]

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v}\right] = \rho\mathbf{b} - \nabla p + \left(\eta_B + \eta/3\right)\nabla(\nabla.\mathbf{v}) + \eta\nabla^2\mathbf{v} \qquad (2.1)$$

where η is the shear viscosity, η_B is the bulk viscosity and **b** is the body force per unit mass . The shear viscosity defined as the tangential stress per unit tangential velocity gradient is a measure of the diffusion of momentum by molecules from regions of fluid possessing higher velocities to adjoining regions possessing lower velocities, while the bulk viscosity defined as the normal stress per unit normal velocity gradient is a measure of the resistance offered by a fluid to pure compression or dilation [4]. According to kinetic theory the bulk viscosity arises because the kinetic energy of molecules is transferred to the internal degrees of freedom, and is proportional to the characteristic (or relaxation) time during which the transfer of energy takes place [5].

Using the vector identity $$\nabla \mathrm{x} \nabla \mathrm{x} \mathbf{v} = -\nabla^2\mathbf{v} + \nabla(\nabla.\mathbf{v}), \qquad (2.2)$$

the Navier-Stokes equation (2.1) can be re-written as

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{v}\right] = \rho\mathbf{b} - \nabla p + \left(\eta_B + 4\eta/3\right)\nabla(\nabla.\mathbf{v}) - \eta\nabla \mathrm{x} \nabla \mathrm{x} \mathbf{v} \qquad (2.3)$$

In adapting the Navier-Stokes equation to acoustic wave phenomenon it is appropriate to admit compressibility and discard circulation, which concepts are represented by the divergence and curl terms (of the velocity) respectively. Thus the acoustic wave condition, to be substituted in the general equation of motion, is

$$\begin{aligned} &\nabla.\mathbf{v} \neq 0 \\ &\nabla \mathrm{x} \nabla \mathrm{x} \mathbf{v} = 0 \end{aligned} \qquad (2.4)$$

where **v** is understood to be a small variation in particle velocity introduced by the acoustic wave.

For a propagating acoustic wave in a source free medium, the term which represents the body force will not apply as it denotes an external force. While the gravitational force is always present, any acoustic wave with wavelength λ much smaller than c^2/g, where c is defined in (5.2) and g is the acceleration due to gravity, is negligibly affected by gravity [6]. When the wave is of small amplitude, we ignore non-linear terms and, the equation of motion for an acoustic wave in a homogeneous Newtonian fluid reduces to,

$$\rho\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta_a\nabla(\nabla.\mathbf{v}),\ \eta_a = \eta_B + 4\eta/3, \qquad (2.5)$$

an expression similar to the Navier-Stokes equation for irrotational flow in a compressible Newtonian fluid. In (2.5) we have used Stokes linearization [7]

$$(\mathbf{v}.\nabla)\mathbf{v} = 0. \qquad (2.6)$$

3.0 Equation of continuity

The relationship between the motion of the fluid and its compression or dilation is given by the equation of continuity [8],

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0, \qquad (3.1)$$

which is in general non-linear. However, for small amplitude waves we define the condensation s,

$$s = (\rho - \rho_o)/\rho_o, \qquad (3.2)$$

where s is a very small quantity and ρ_o is a constant equilibrium density of the fluid. In terms of the condensation, the equation of continuity may be re-written as

$$\rho_o \frac{\partial s}{\partial t} + \rho_o \nabla.\mathbf{v} + \rho_o (\mathbf{v}.\nabla s + s\nabla.\mathbf{v}) = 0 . \tag{3.3}$$

For small amplitude waves we may ignore non-linear terms, and the continuity equation reduces to

$$\frac{\partial s}{\partial t} + \nabla.\mathbf{v} = 0 . \tag{3.4}$$

Assuming an equation of state for the fluid, the linear continuity equation can be written in terms of pressure and particle velocity.

4.0 Equation of state

The relationship between the internal restoring force and the corresponding deformation in a medium is given by the equation of state. In a fluid the (thermal) equation of state links the pressure, density (or condensation) and temperature. For an acoustic wave in a lossless medium, the pressure and condensation are in phase. One way to account for acoustic energy loss in a medium is to allow a time-delay between the introduction of a sudden pressure change and the attainment of the resulting equilibrium condensation. This is the approach used in the Stokes equation of state, which introduces a relaxation time associated with a particular loss mechanism.

The compression of a fluid causes a rise in its temperature and its dilation is accompanied by a temperature decrease, unless allowance is given for heat exchange. As an acoustic wave propagates through a fluid, the regions that are compressed at any instant are slightly warmer than those which are dilated. In ordinary circumstances, the wavelength is too large and the thermal conductivity too small for any appreciable amount of heat to flow. Thus acoustic wave propagation in a fluid is an adiabatic process.

Given that we intend to use a lossless continuity equation, we are interested in a lossless equation of state. For fluids other than a perfect gas, the adiabatic equation of state is often approximated using [1]

$$p = Bs \tag{4.1}$$

where B is the adiabatic bulk modulus of the fluid, though, in liquids an alternative is the Tait equation of state. However, we assume that the adiabatic equation of state is applicable to an acoustic wave in a fluid.

5.0 Acoustic wave equation

We describe the linearized acoustic wave equation in a viscous fluid using the two coupled equations:

$$\rho_o \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta_a \nabla (\nabla.\mathbf{v}) \tag{5.1a}$$

$$\frac{\partial p}{\partial t} = -B\nabla.\mathbf{v} \tag{5.1b}$$

where ρ_o is required by the approximation leading to (3.4) and p is taken to be the acoustic pressure (that is, a variation in pressure introduced by the acoustic wave). Equation (5.1b) is obtained by substituting (4.1) in the linear equation of continuity (3.4). The two coupled equations in (5.1) is a system of four partial differential equations governing four unknown scalar field functions; namely, three velocity components and the pressure. The acoustic wave equation representation (5.1) is suitable for finite difference time domain discretization.

The acoustic wave equation can be expressed, as a single partial differential equation, in terms of a single field function by using the two coupled equations in (5.1). Differentiating (5.1a) with respect to time and using (5.1b) to eliminate p leads to the acoustic wave equation in a viscous fluid in terms of the particle velocity:

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - c^2 \nabla^2 \mathbf{v} = \frac{\eta_a}{\rho_o} \frac{\partial}{\partial t} [\nabla(\nabla.\mathbf{v})] , c^2 = \sqrt{\frac{B}{\rho_o}} , \tag{5.2}$$

where c is the thermodynamic speed of sound in the medium. Observe that in equation (5.2) the acoustic energy loss in the fluid is represented using a term that involves the viscosities, and so the loss term is zero in an inviscid fluid (η, $\eta_B = 0$), with the acoustic wave equation in an inviscid fluid reducing to the usual expression:

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - c^2 \nabla^2 \mathbf{v} = 0 . \qquad (5.3)$$

Also note that equation (5.2) is written in symbolic notation. Specifically the x-component of (5.2) in 3-D rectangular coordinates is

$$\frac{\partial^2 v_x}{\partial t^2} - c^2 \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) = \frac{\eta_a}{\rho_o} \frac{\partial}{\partial t} \left[\frac{\partial}{\partial x} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \right] \qquad (5.4)$$

which in 1-D reduces to

$$\frac{\partial^2 v_x}{\partial t^2} - c^2 \frac{\partial^2 v_x}{\partial x^2} = \frac{\eta_a}{\rho_o} \frac{\partial}{\partial t} \left(\frac{\partial^2 v_x}{\partial x^2} \right). \qquad (5.5)$$

The significance of equation (5.5) is that once the particle velocity varies with position acoustic wave propagation in a 1-D viscous fluid involves energy loss. This is the case because viscous forces come into play when there is a velocity gradient, irrespective of whether the velocity gradient is ‘tangential’ or ‘normal’ [9].

6.0 Dispersive and dissipative characteristics in 1-D

We examine the dispersive and dissipative characteristics of the medium by assuming that the solution of equation (5.5) is simple harmonic with the exponential form

$$v_x = v_o exp - i(\omega t - kx) , \qquad (6.1)$$

where ω is angular frequency and k is wave number. By substituting (6.1) in (5.5) we have the 1-D dispersion relation for the medium

$$\left(c^2 - i\,\omega\eta_a / \rho_o\right)k^2 = \omega^2 . \qquad (6.2)$$

The dispersion relation (6.2) implies a complex wave number ($k = \alpha + i\beta$) with a real propagation constant

$$\alpha = \frac{\omega}{c\sqrt{2}} \frac{\left(\sqrt{1 + \left(\omega\eta_a / c^2\rho_o\right)^2} + 1 \right)^{1/2}}{\sqrt{1 + \left(\omega\eta_a / c^2\rho_o\right)^2}} \qquad (6.3)$$

and a damping coefficient

$$\beta = \frac{\omega}{c\sqrt{2}} \frac{\left(\sqrt{1 + \left(\omega\eta_a / c^2\rho_o\right)^2} - 1 \right)^{1/2}}{\sqrt{1 + \left(\omega\eta_a / c^2\rho_o\right)^2}} . \qquad (6.4)$$

The dependence of the propagation constant (α) on the angular frequency means that the wave phase velocity (ω/α) is a function of the frequency and the medium is dispersive. Also, the presence of a non-zero damping coefficient defines a dissipative medium. Observe from equation (6.4) that if the viscosities are zero then the damping coefficient is zero.

In the limit $\omega\eta_a/\rho_o c^2 << 1$, the acoustic wave absorption coefficient (6.4) in the fluid reduces to

$$\beta = \eta_a \omega^2 / 2\rho_o c^3 . \qquad (6.5)$$

Furthermore the 1-D case implies that we are limited to plane waves, i.e. a wave which depends on time and a single space coordinate, and all quantities in such a wave are independent of the other directions, say, y and z.

Thus we may only expect normal stresses and by definition only the bulk viscosity will apply. Consequently the viscosity term in (2.5) and in the subsequent equations becomes $\eta_a = \eta_B$, and equation (6.5) reduces to

$$\beta = \eta_B \omega^2 / 2\rho_o c^3 \ . \tag{6.6}$$

Following the definitions of the viscosity coefficients as explained in [4] the bulk viscosity is a fundamental contributor to energy loss in 1-D acoustic wave propagation in fluids (especially, liquids).

7.0 2-D dispersion relation

Two-dimensional acoustic wave propagation in a viscous fluid, in rectangular coordinates, satisfies the coupled pair of partial differential equations:

$$\begin{aligned}\frac{\partial^2 v_x}{\partial t^2} - c^2\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) &= \frac{\eta_a}{\rho_o}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right] \\ \frac{\partial^2 v_y}{\partial t^2} - c^2\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) &= \frac{\eta_a}{\rho_o}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)\right]\end{aligned} \tag{7.1}$$

For a plane wave with the propagation vector parallel to the z = 0 plane the acoustic field depends on (x, y, t), thus

$$v_x , v_y \propto exp - i(\omega t - k_x x - k_y y) . \tag{7.2}$$

Using equation (7.2) we can re-write (7.1) in matrix form:

$$\begin{bmatrix} c^2\left(k_x^2 + k_y^2\right) - \omega^2 + i\omega\eta_a k_x^2 / \rho_o & -i\omega\eta_a k_x k_y / \rho_o \\ -i\omega\eta_a k_x k_y / \rho_o & c^2\left(k_x^2 + k_y^2\right) - \omega^2 + i\omega\eta_a k_y^2 / \rho_o \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} = 0 \tag{7.3}$$

We derive the 2-D dispersion relation by setting the determinant of the square matrice in (7.3) to zero. This gives the 2-D dispersion relation

$$\omega^4 + c^4\left(k_x^2 + k_y^2\right)^2 - 2\omega^2 c^2\left(k_x^2 + k_y^2\right) = -i\frac{\omega\eta_a}{\rho_o}\left(k_x^2 + k_y^2\right)\left[c^2\left(k_x^2 + k_y^2\right) - \omega^2\right]. \tag{7.4}$$

Observe that equating either k_x or k_y to zero in equation (7.4) does not reduce the 2-D dispersion relation to the 1-D dispersion relation (6.2), because of the ‘coupling terms’ in equation (7.1) which are not present in (5.5). These coupling terms imply a shearing stress (according to Newton’s law of viscosity) which is not applicable in one-dimension. The term that represents the viscous force in the 1-D situation denotes a non-shearing (normal) stress. Thus acoustic wave propagation in a 2-D viscous fluid involves both shearing and non-shearing viscous forces while propagation in a 1-D viscous fluid involves only a non-shearing viscous force.

8.0 Conclusion

The formulation presented above for acoustic wave propagation in a viscous Newtonian fluid, which makes use of the Navier-Stokes momentum equation, hinges on the acoustic wave condition, namely, the requirement of a compressible and irrotational fluid. Essentially, the variation in particle velocity introduced by the presence of the acoustic wave should have a divergence and no circulation. This leads to an acoustic wave equation that includes an energy loss term arising from a combination of shearing and non-shearing viscous forces. In a liquid or for a plane acoustic wave the viscous loss is due mainly to a non-shearing viscous force.

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