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Effects of damping and exponentially decaying foundation on the motions of finite thin beam subjected to travelling loads

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Abstract

The dynamic response of a finite elastic thin beam to fast moving heavy concentrated forces is investigated. The beam is assumed to be under tensile stress and have simple supports at both ends. Furthermore, the beam is assumed to rest on elastic foundation of the exponential rigidity and the moving concentrated forces is assumed to move with constant velocity type of motion. The fourth order partial differential equation governing the flexural motions of the elastic systems is solved using mode superimposition and Integral transform method and the closed-form solutions to this beam problem is obtained. Effects of some beam parameters on the response of the beam are classified. Results presented both analytically and numerically in this paper are readily applicable in engineering design and analysis and for further investigation in structural dynamics.

Keywords**:** Dynamic response, Concentrated forces, Exponential rigidity, Flexural motions, Beam parameters

1.0 Introduction

This paper scrutinizes the practical engineering problem of the forced vibrations of prestressed finite beam under the actions of fast traveling concentrated forces. Applications of this class of problems are enormous. It is applicable in studying electromagnetic phenomenon that arose from electrical machinery, communications equipment and computer chips. Applications also includes the response of railroad rails to moving trains, the response of bridges and elevated roadways to moving vehicles, machine chain and belt drives, computer tape drives, floppy disks and video cassette recorders all these and many more examples are indicated in [1-10].

Generally, it is well known that structural vibration problems are modeled either as a moving force problems or moving mass problems. There exist very large bodies of literature devoted to these classes of problems few example of these can be seen in [10-15]. When fast traveling heavy loads traverses on a structure, its effect on such structural members are dual, on one hand is the gravitational effects of the moving load while on other hand are the inertia effects of the moving mass. The former is termed the moving force problem while the later is termed the moving mass problem.

Structural dynamics problems (whether moving force or moving mass) have been modeled with or without foundation. Few examples of beam models in literatures in which the elastic structures subjected to traveling loads were not placed on elastic foundation can be seen in [16-20]. It is note worthy however, that elastic solid structures (Membrane, Rods, Beams, Plates or shells) on elastic foundation has a lot of practical applications in Mathematical physics, Applied Mathematics and Engineering. In particular, structural members on elastic foundations are commonly used in several applications such as roadways, runways, rail road tracks, submerged pipes etc.

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Studies on elastic structures resting on elastic foundation are numerous in literatures references [21-27] presented handy examples of such problems. In all these studies nevertheless, elastic foundations of constant and linear rigidity has been the form of the foundations commonly employed. Analysis of beams on non-linear elastic foundation is not common in literatures. To the best of authors knowledge, the dynamical system involving the problem of the prestressed damped elastic beam resting on exponentially decaying foundation and subjected to fast traveling loads does not exist in literature.

Thus, this paper assesses the dynamic response of prestressed finite damped thin beam resting on non-uniform elastic foundation (whose rigidity is of exponential type) and under the actions of fast traveling heavy concentrated forces. The specific objective is to obtain a closed-form solution to this beam problem and to classify the effects of some beam parameters namely; axial force, foundation moduli and internal damping on the vibrating system.

2.0 The mathematical formulation

Consider the motion of a uniform simply supported finite beam under tensile stress which rest on elastic foundation of the exponential rigidity. The concentrated forces traversing on the beam is assumed to move with constant velocity type of motion. According to the simple beam theory of flexures, the equation of motion with damping included of the prestressed elastic thin beam is given by the fourth order partial differential equation

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu\frac{\partial^2 V(x,t)}{\partial x^2} + \varepsilon_0\frac{\partial V(x,t)}{\partial t} + K(x)V(x,t) = P(x,t) \qquad (2.1)$$

where

EI is the flexural rigidity, $V(x,t)$ is the transverse displacement response of the vibrating beam, x is the spatial coordinate, t is the time, N is the axial force, μ is the mass per unit length, ε_0 is the damping coefficient, K(x) is the non-uniform elastic foundation function and P(x,t) is the traversing load.

Since the beam is assumed to have simple support at both ends, the boundary conditions are thus given as

$$V(0,t) = 0 = V(L,t), \quad \frac{\partial V(0,t)}{\partial x} = 0 = \frac{\partial^2 V(L,t)}{\partial x^2} \qquad (2.2)$$

and the initial conditions are

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t} \qquad (2.3)$$

In this study, the traversing load is assumed to be of constant magnitude so that

$$P(x,t) = P\delta(x - c_m t) \qquad (2.4)$$

c_m is the velocity of the mth particle of the system and we shall use elastic foundation of exponential rigidity given by

$$K(x) = K_0 e^{-\lambda x} \qquad (2.5)$$

where λ is a constant and K_0 is the foundation constant.

Using (2.4) and (2.5) in equation (2.1) we have,

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu\frac{\partial^2 V(x,t)}{\partial x^2} + \varepsilon_0\frac{\partial V(x,t)}{\partial t} + K_0 e^{-\lambda x}V(x,t) = P\delta(x - v_i t) \quad (2.6)$$

In what follows, solution to the fourth order partial differential equation (2.6) governing the motion of the thing beam under the actions of concentrated moving forces is sought. This beam problem can be handled analytically or numerically, but analytical technique is desirable as the solution so obtained sheds more light on some vital information about the vibrating system.

3.0 The solution techniques

In this paper, in order to compute the transverse deflection $V(x,t)$ of the vibrating beam, use is made of mode superimposition technique. By this technique the transverse deflection of the beam can be written as

$$V(x,t) = \sum_{m=1}^{\infty} Y_m(t) U_m(x) \qquad (3.1)$$

where $Y_m(t)$ are coordinates in modal space and $U_m(x)$ are the normal modes of free vibration written as

$$U_m(x) = \sin \lambda_m x + A_m \cos \lambda_m x + B_m \sinh \lambda_m x + C_m \cosh \lambda_m x \qquad (3.2)$$

where the constants A_k, B_k and C_k define the shape and amplitude of the beam vibration. Their values depend on the boundary condition associated with the structure. For beams with simple supports, it can be shown that $A_k = B_k = C_k = 0$ and $\lambda_m = \frac{m\pi}{L}$. Thus the transverse deflection of a simply supported elastic beam, using an assumed mode method and taking into account equation (3.2) can be written as

$$V(x,t) = \sum_{m=1}^{\infty} Y_m(t) \sin \frac{m\pi x}{L} \qquad (3.3)$$

Substituting equation (3.3) into the governing equation (2.6) and after some simplifications and rearrangements we obtain

$$\sum_{m=1}^{\infty} \left\{ EI\left(\frac{m\pi}{L}\right)^4 Y_m(t) \sin \frac{m\pi x}{L} + \mu \ddot{Y}_m(t) \sin \frac{m\pi x}{L} + N\left(\frac{m\pi}{L}\right)^2 \dot{Y}_m(t) \sin \frac{m\pi x}{L} \right.$$
$$\left. + \varepsilon_0 \dot{Y}_m \sin \frac{m\pi x}{L} + K_0 e^{-\lambda x} Y_m(t) \sin \frac{m\pi x}{L} \right\} - P\delta(x - c_m t) = 0 \qquad (3.4)$$

The solution technique requires that the RHS of equation (3.4) be orthogonal to the function $\sin\frac{k\pi x}{L}$. Thus, multiplying equation (3.4) by $\sin \frac{k\pi x}{L}$ and integrating from 0 to L with respect to *x* after some simplifications and rearrangements yield

$$\sum_{m=1}^{\infty} \left(A_1 Y_m(t) + A_2 \ddot{Y}_m(t) + A_3 Y_m(t) + A_4 \dot{Y}_m(t) + A_5 Y_m(t) \right) = \int_0^L P\delta(x - c_m t) \sin \frac{k\pi x}{L} dx \qquad (3.5)$$

where, $A_1 = EI\left(\frac{m\pi}{L}\right)^4 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx$, $A_2 = \mu \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx$

$A_3 = N\left(\frac{m\pi}{L}\right)^2 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx$, $A_4 = \varepsilon_0 \int_0^L \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx$

and $A_5 = K_0 \int_0^L e^{-\lambda x} \sin \frac{m\pi x}{L} \sin \frac{k\pi x}{L} dx \qquad (3.6)$

we note that the dirac delta function has the property that

$$\int_h^L \delta(x - q) f(x) dx = \begin{cases} 0, & q < h \\ f(q), & h < q < L \\ 0, & q > h \end{cases} \qquad (3.7)$$

Thus in view of equation (3.7), equation (3.5) can be written after some rearrangements as

$$\sum_{m=1}^{\infty}\left(\ddot{Y}_m(t)+B_1\dot{Y}_m(t)+B_2Y_m(t)\right)=B_3\sin\frac{k\pi c_m t}{L} \quad (3.8$$

where, $$B_1=\frac{A_4}{A_2},\ B_2=\frac{A_1+A_3+A_5}{A_2} \text{ and } B_3=\frac{P}{A_2} \quad (3.9)$$

Now considering only the mth particle of the dynamical system we have

$$\ddot{Y}_m(t)+B_1\dot{Y}_m(t)+B_2Y_m(t)=B_3\sin\frac{k\pi c_m t}{L} \quad (3.10)$$

Subjecting the second order ordinary differential equation (3.10) to a Laplace transform

$$\left(\tilde{\cdot}\right)=\int_0^{\infty}(\cdot)e^{-st}dt \quad (3.11)$$

in conjunction with the initial conditions defined in (2.3), yields the following algebraic equation

$$\left(s^2+B_1s+B_2\right)Y_m(s)=B_3\frac{\left(\frac{j\pi c_m}{L}\right)}{s^2+\left(\frac{j\pi c_m}{L}\right)^2} \quad (3.12)$$

which after some simplifications and rearrangements yields

$$Y_m(s)=\frac{B_3}{\alpha_1-\alpha_2}\left(\frac{1}{\alpha_1}\frac{\beta}{s^2+\beta^2}\cdot\frac{\alpha_1}{s-\alpha_1}-\frac{1}{\alpha_2}\frac{\beta}{s^2+\beta^2}\cdot\frac{\alpha_2}{s-\alpha_2}\right) \quad (3.13)$$

where

$$\beta=\frac{j\pi c}{L},\ \alpha_1=\frac{-B_1+\sqrt{\left(B_1^2-4B_2\right)}}{2} \text{ and } \alpha_1=\frac{-B_1-\sqrt{\left(B_1^2-4B_2\right)}}{2} \quad (3.14)$$

In order to obtain the Laplace inversion of equation (3.13), we shall adopt the following representations

$$g(s)=\frac{\beta}{s^2+\beta^2},\ f_1(s)=\frac{\alpha_1}{s-\alpha_1} \text{ and } f_2(s)=\frac{\alpha_2}{s^2-\alpha_2} \quad (3.15)$$

So that the Laplace inversion of equation (3.15) is the convolution of f_i*'s* and *g* defined as

$$f_i*g=\int_0^t f_i(t-u)g(u),i=1,2 \quad (3.16)$$

Thus the Laplace inversion of equation (3.13) is given by

$$Y_m(t)=\frac{B_3}{(\alpha_1-\alpha_2)}\left[\frac{1}{\alpha_1}\cdot I_a-\frac{1}{\alpha_2}\cdot I_b\right] \quad (3.17)$$

where $$I_a=\int_0^t e^{\alpha_1(t-u)}\sin\beta u du \text{ and } I_b=\int_0^t e^{\alpha_2(t-u)}\sin\beta u du \quad (3.18)$$

Thus in view of equation (3.17) taking into account integrals (3.18) we have

$$Y_m(t)=\frac{B_3\alpha_1}{(\alpha_1-\alpha_2)\left(\alpha_1^2+\beta^2\right)}\left[\frac{\beta}{\alpha_1}\left(e^{\alpha_1t}-\cos\beta t\right)-\sin\beta t\right]-\frac{B_3\alpha_2}{(\alpha_1-\alpha_2)\left(\alpha_2^2+\beta^2\right)}\left[\frac{\beta}{\alpha_2}\left(e^{\alpha_2t}-\cos\beta t\right)-\sin\beta t\right] \quad (3.19)$$

Using the expression (3.19) in equation (3.1) we have

$$V(x,t)=\sum_{m=1}^{\infty}\frac{B_3\alpha_1}{(\alpha_1-\alpha_2)\left(\alpha_1^2+\beta^2\right)}\left[\frac{\beta}{\alpha_1}\left(e^{\alpha_1t}-\cos\beta t\right)-\sin\beta t\right]-\frac{B_3\alpha_2}{(\alpha_1-\alpha_2)\left(\alpha_2^2+\beta^2\right)}\left[\frac{\beta}{\alpha_2}\left(e^{\alpha_2t}-\cos\beta t\right)-\sin\beta t\right]\sin\frac{m\pi x}{L} \quad (3.20)$$

which represents the transverse displacement response of prestressed damped thin beam resting on exponentially decaying foundation and subjected to fast moving forces.

It can be shown, following the same procedures as those used in [19] that the series solution (3.20) converges rapidly.

4.0 Discussion of analytical solution

In any study concerning a vibrating system, resonance phenomenons is of great concern to researchers or in particular, design engineers. Because, the transverse deflection of elastic beams subjected to fast traveling loads may grow without bound. It is clearly seen from equation (3.18) above that a prestressed damped thin beam resting on exponentially decaying foundation and subjected to fast moving loads will experience resonance effects whenever

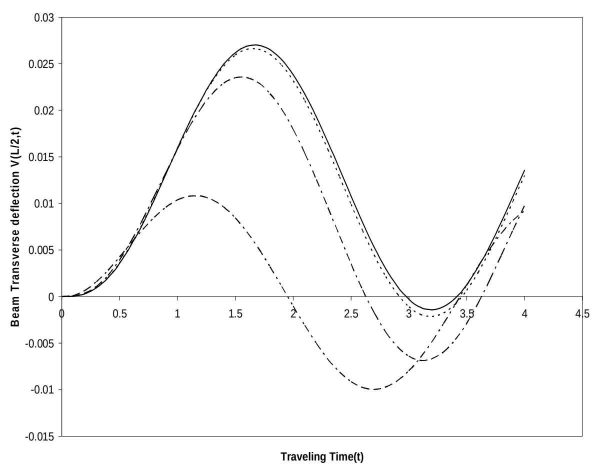
$$\alpha_1^2 = -\beta^2 \text{ or } \alpha_2^2 = -\beta^2 \tag{4.1}$$

and the speed at which this occurs is termed the critical speed and is given by

$$c_m^2 = \left(4B_2 + 2B_1\sqrt{B_1^2 - 4B_2} - 2B_1^2\right)\cdot\left(\frac{L}{j\pi}\right)^2 \tag{4.2}$$

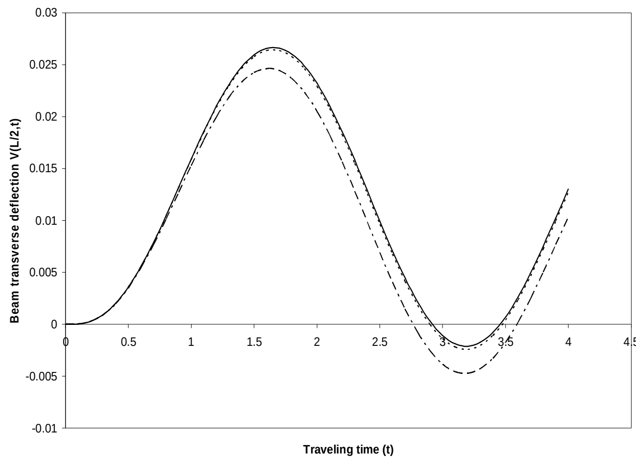
5.0 Illustrative examples

To illustrate the theory proposed in this paper numerically, the velocity of the fast moving concentrated loads is taken to be 8.128m/s where the span L of the beam is taken to be 12.192m. The value of flexural rigidity *EI* is 6068242, the values of foundation moduli are varied between 4000 N/m^3 and 400000 N/m^3, the values of axial force N are varied between 20000 and 2000000 N. In figure 5.1, the transverse displacement response of a prestressed finite thin beam resting on exponentially decaying foundation and under the actions of traveling concentrated forces is displayed. It is clearly seen that when the value of foundation moduli K_0 is fixed and for fixed value of the damping coefficient ε_0, the displacements of a prestressed finite thin beam resting on exponentially decaying foundation and traversed by concentrated moving forces decreases as the values of axial force *N* increases.



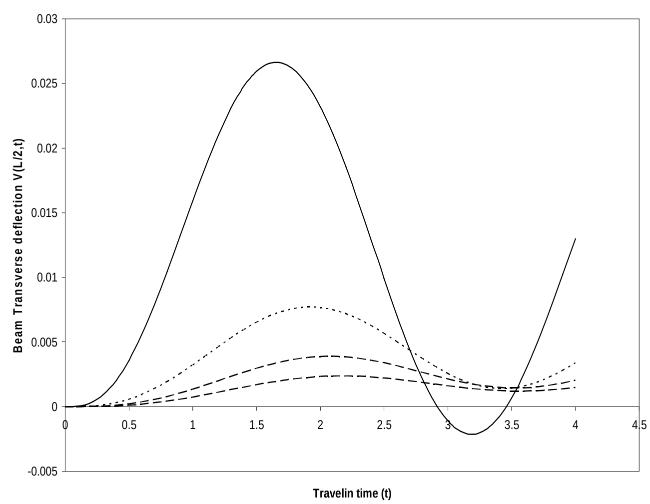
____ N = 0, …. *N* = 20000, _ _ _ _*N* = 200000, …… *N* = 2000000

Figure 5.1: The transverse displacement response of a simply supported damped beam resting on exponentially decaying foundation and subjected to moving forces for various values of axial force *N* and for fixed values of foundation moduli K_0 = 40000 and damping coefficient ε_0 = 76



____ K = 0, …… K = 400000, _ _ _ _ K = 4000000

Figure 5.2: The deflection profile of a simply supported damped beam resting on exponentially decaying foundation and subjected to moving forces for various values of foundation moduli K_0 and for fixed values of axial force N = 20000 and damping coefficient $\varepsilon_0 = 76$



____ $\varepsilon_0 = 76$, …… $\varepsilon_0 = 236$, ----- $\varepsilon_0 = 393$, ….. $\varepsilon_0 = 550$

Figure 5.3: The response amplitude of a simply supported damped beam resting on exponentially decaying foundation and subjected to moving forces for various values of damping coefficient ε_0 and for fixed values of foundation moduli K_0=40000 and axial force N=20000

Figure 5.2 Depicts the deflection profile of a prestressed finite thin beam resting on exponentially decaying foundation and under the actions of concentrated forces and is shown from the figure that as the values of foundation moduli K_0 increases, for fixed values of axial force N and damping coefficient ε_0 , the response amplitudes of the beam decreases.

Furthermore, the response amplitude of a prestressed finite thin beam resting on exponentially decaying foundation and under the actions of moving concentrated forces is shown in figure 5.3. It is deduced from the figure that for fixed values of axial force N and foundation moduli K_0, the amplitudes of the damped elastic beam decreases as the values of the damping coefficient ε_0 increases

6.0 Concluding remarks

In this study, the dynamic response of a prestressed damped beam resting on exponentially varying magnitude foundation to fast traveling loads has been investigated. The fourth order partial differential equation governing the motion of the elastic beam is handled using an assumed mode technique and the method of integral transform and a closed form solution of the beam problem is obtained. The advantage of these techniques is that solution so obtains shed light on vital information about the vibrating system. Analytical and numerical results show that the higher the values of foundation stiffness K_0 the lower the deflection profile of the thin beam. Similarly, as we increase the value of axial force N the transverse displacement response of the elastic beam increases. This new results are in perfect agreement with existing results. Furthermore, as the value of damping coefficient increases, the amplitude of the vibrating beam reduces. Finally, it is found that as we increase the values of foundation stiffness K_0, axial force N and damping coefficient ε_0 the critical speed of the vibrating system involving prestressed elastic thin beam under the actions of concentrated moving forces increases and the risk of resonance is sufficiently reduced.

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