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Time-dependent random forcing and buckling of a damped finite column on non-linear elastic foundations

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Abstract

This paper is concerned with time-dependent random vibrations and dynamic buckling of a finite deterministically imperfect column resting on "softening" nonlinear (cubic) elastic foundations but trapped by a stochastically random, zero-mean, time-dependent load. The random load is imbued with certain statistical characterizations including a correlated exponential cosine autocorrelation which confers some form of randomness on the response statistic. Assuming the mean square lateral displacement to be a suitable statistical characterization of the random response, the dynamic buckling load of the nonlinear structure is determined asymptotically via regular perturbation procedures. It is found out that the dynamic buckling load is of order $R_0^{-\frac{1}{2}}$, where R_0 is the variance of the random load. It is also established that the value of the dynamic buckling load, in the case where the buckling modes are strictly in the shape of imperfection, is less than its value in a similar case where the buckling modes are not strictly in the shape of imperfection.

1.0 Introduction

Stochasticity in respect of the stability of elastic structures has, over the years, been discussed in connection with geometrical imperfections, but seldom with respect to the time-dependent loading history. Such were the cases in earlier investigations such as those by Elishakoff [1], Amazigo [2,3] , Boyce [4] , Fraser [5] and Hansen and Roorda [6] ,among others. While most of the cited investigations addressed the subject matter from the static point of view, Spanos et al [7] delved into the realm of dynamical consideration in a heuristic spectral estimate of bivariate non-stationary process. Worthy of mention is the work of Lin and Cai [8] where investigations of stochasticity from the dynamical point of view were also given some level of prominence. In general, investigations on stochasticity in relation to strictly time-dependent random loading histories in the landscape of dynamic buckling are rare. In this investigation, we discuss stochasticity in a dynamical system involving the dynamic stability of a damped finite simply-supported and time-dependent randomly loaded column in which a random vibration is produced in response to a deterministic dynamical system.

2.0 Formulation

The dimensional equation satisfied by the lateral displacement $W(X,T)$ of a damped finite simply-supported column lying on a deterministically imperfect non-linear (cubic) elastic foundation [9] is

$$m_0 W,_{TT} + c_0 W,_T + EIW,_{XXXX} + + P(T)W,_{XX} + k_1 W - \alpha k_3 W^3 = -P(T)\frac{d^2\overline{W}}{dX^2} \qquad (2.1)$$

where m_0 is the mass per unit length, c_0 is the damping constant per unit length per velocity, EI is the bending

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stiffness , $P(T)$ is a zero-mean stationary Gaussian load function of the univariate time variable T and of known autocorrelation $R_f(t)$, E and $I$are the Young's modulus and moment of inertia respectively , $\overline{W}(X)$ is a stress-free time-independent twice-differentiable imperfection function of the space variable X; k_1 and k_3 are constants such that $k_1 > 0$, $k_3 > 0$, while α is the imperfection-sensitivity parameter which is such that if α=1, the non-linear elastic foundation is said to be a "softening spring" where as if α = -1, the non-linear foundation is said to be a " hardening spring" .The finite column rests on a nonlinear elastic foundation that produces a restoring force per unit length of $k_1W - \alpha k_3W^3$. Since we are considering a column resting on a "softening" foundation, we shall automatically set α =1. In this formulation, we shall neglect axial inertia as well as nonlinearities higher than the cubic. A subscript following a comma shall indicate partial differentiation and homogeneous initial conditions are assumed .We now assume the following non-dimensional quantities:

$$x=\left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X\,,\ w=\left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W\,,\ \lambda f(t)=\frac{P(T)}{2(EIk_1)^{\frac{1}{2}}}\,,\ t=\left(\frac{k_1}{m_0}\right)^{\frac{1}{2}} T,\ \xi\,\overline{w}=\left(\frac{k_3}{k_1}\right)^{\frac{1}{2}}\overline{W} \quad (2.2a)$$

$$2c=c_0\left(\frac{1}{m_0k_1}\right)^{\frac{1}{2}} \quad (2.2b)$$

3.0 Solution of the problem

We assume $0 < c < 1$, $0 < \lambda < 0$, 0, $0 < \xi << 1$, so that the non-dimensional form of equation (2.1), together with the homogeneous initial conditions, becomes

$$w,_{tt}+2c\,w,_t+w,_{xxxx}+2\lambda\, f(t)w,_{xx}+w-w^3=-2\lambda f(t)\xi\frac{d^2\overline{w}}{dx^2}\ ,\ t>0,\ 0<x<\pi, \quad (3.1a)$$

$$w=w,_{xx}=0 \text{ at } x=0,\ \pi,\ t\geq 0\ ,\ \ w(x,0)=w,_t(x,0)=0,\ 0<x<\pi \quad (3.1b)$$

Here, λ is the amplitude of the random load $f(t)$ while ξ is the amplitude of imperfection function $\overline{w}$. We stress that $f(t)$ is a non-dimensional zero–mean Gaussian random load function of the univariate time variable t with correlated exponentially decaying harmonic autocorrelation $R_f(t)$, given by

$$R_f(\tau)=\langle f(t)f(t+\tau)\rangle=R_0e^{-\alpha|\tau|}\cos\Omega\tau\ ,0<\alpha<1,\ 0<\Omega\ ,0<R_0<1 \quad (3.2a)$$

where $R_f(0) = R_0$, is the variance of the load $f(t)$ and $\langle\Lambda\rangle$, is the Mathematical expectation defined as follows :

$$\langle\Lambda\rangle=Lim\ \hat{T}\to\infty\ \left\{\frac{1}{\hat{T}}\int_0^{\hat{T}}(\Lambda)\,dt\right\} \quad (3.2b)$$

In our quest for solution, we are to determine a particular value of the load amplitude λ, designated as λ_D, called the dynamic buckling load, for which the damped structure buckles dynamically. We define λ_D as the largest load amplitude for which the problem (3.2a,b) has a bounded solution for all time $t > 0$.

For solution, we let $\in=\lambda\xi$, $0<\in<1$,and with this, we recast (3.1a) vividly as

$$w,_{tt}+2c\,w,_t+w,_{xxxx}+\frac{2\in f(t)w,_{xx}}{\xi}+w-w^3=-2f(t)\in\frac{d^2\overline{w}}{dx^2}\ ,\ t>0,\ 0<x<\pi \quad (3.3)$$

We shall now solve (3.3), using (3.1b) and (3.2b) and so we let

$$w(x,t)=\sum_{i=1}^{\infty}w^{(i)}(x,t)\in^i \quad (3.4)$$

and substitute same into (3.3) and (3.1b) and equate the coefficients of $\in^i$, $i=1,2,3,\Lambda$ to get the following equations

$$Lw^{(1)} \equiv w^{(1)}_{,tt} + 2\,\mathrm{c}w^{(1)}_{,t} + w,_{xxxx} + w^{(1)} = -2f(t)\frac{d^2\overline{w}}{dx^2} \tag{3.5}$$

$$Lw^{(2)} = -\frac{2f(t)\,\mathrm{w}^{(1)}_{,xx}}{\overline{\xi}} \tag{3.6}$$

$$Lw^{(3)} = -\frac{2f(t)\,\mathrm{w}^{(2)}_{,xx}}{\overline{\xi}} + w^{(1)\,3} \tag{3.7}$$

$$w^{(i)} = w^{(i)}_{,xx} = 0 \text{ at } x = 0,\pi\,;\ \mathrm{w}^{(i)}(x,0) = w^{(i)}_{,xx}(x,0) = 0\,,\ \mathrm{i} = 1,2,3,\Lambda \tag{3.8}$$

Based on the boundary conditions in (3.8), we let $\overline{w} = \overline{a}_m \sin\ mx$ (3.9)

To solve equations (3.5) - (3.9), we let

$$w^{(i)}(x,t) = \sum_{n=1}^{\infty} A_n(t)\sin \mathrm{nx} \tag{3.10}$$

We substitute (3.9) and (3.10) into (3.5), for $i = 1$, and get

$$\sum_{n=1}^{\infty}\left[A^{(1)}_{n,tt} + 2cA^{(1)}_{n,t} + \left(n^4+1\right)\mathrm{A}^{(1)}_{\mathrm{n}}\right]\sin \mathrm{nx} = 2\overline{a}_m m^2 f(t)\sin \mathrm{mx} \tag{3.11a}$$

The only non-zero solution of (3.11a) is for the case $n = m$, in which case we have

$$A^{(1)}_{m,tt} + 2cA^{(1)}_{m,t} + \left(m^4+1\right)\mathrm{A}^{(1)}_{\mathrm{m}} = 2\overline{a}_m m^2 f(t) \tag{3.11b}$$

$$^{(1)}_{m}(0) = 0\,,\ \mathrm{A}^{(1)}_{\mathrm{m},t}(0) = 0 \tag{3.11c}$$

On solving (3.11b,c) we get

$$A^{(1)}_m(t) = 2\overline{a}_m m^2\int_0^t h(\tau)f(t-\tau)\,\mathrm{d}\tau\,,\ \mathrm{h}(\tau) = \frac{e^{-c\tau}\sin\varphi_m\tau}{\varphi_m}\ ,\ \varphi_m = \sqrt{m^4+1-c^2} \tag{3.12}$$

Thus, we conclude that

$$w^{(1)}(x,t) = A^{(1)}_m(t)\sin\ mx \tag{3.13}$$

If we substitute for terms on the right hand side of (3.8), we have

$$Lw^{(2)} = \frac{2m^2 f(t)A^{(1)}_m(t)}{\overline{\xi}}\sin\ mx \tag{3.14}$$

On solving (3.14), using (3.10) and (3.8), for i = 2, we get

$$w^{(2)}(x,t) = A^{(2)}_m(t)\sin \mathrm{mx}\,,\ \mathrm{A}^{(2)}_{\mathrm{m}}(t) = \frac{4\overline{a}_m m^4 F(t)}{\overline{\xi}} \tag{3.15(a)}$$

$$\mathrm{F}(\mathrm{t}) = \int_0^t\int_0^{t-\tau_1} h(\tau_1)h(\tau_2)f(t-\tau_1)f(t-\tau_1-\tau_2)\,d\tau_1 d\tau_2 \tag{3.15b}$$

We now substitute into (3.7) for $w^{(1)}$ and $w^{(2)}$ from (3.135) and (3.15a,b) respectively and get

$$Lw^{(3)} = \frac{8\overline{a}_m m^6 \mathrm{f}(\mathrm{t})\,\mathrm{F}(\mathrm{t})\sin \mathrm{mx}}{\overline{\xi}^2} + \frac{A^{(1)\,3}_m}{4}\left(3\sin \mathrm{mx} - \sin 3\mathrm{mx}\right) \tag{3.16}$$

We now solve (3.16) subject to (3.8) and (3.10), for i = 3, and get

$$\sum_{n=1}^{\infty}\left[A^{(3)}_{n,tt} + 2cA^{(3)}_{n,t} + \left(n^4+1\right)A^{(3)}_n\right]\sin \mathrm{nx} = \frac{8\overline{\mathrm{a}}_{\mathrm{m}} m^6 f(t)\,\mathrm{F}(\mathrm{t})\sin \mathrm{mx}}{\overline{\xi}} + \frac{A^{(1)3}_m}{4}\left(3\sin \mathrm{mx} - \sin 3\mathrm{mx}\right) \tag{3.17a}$$

For $n = m$ in (3.17a), we

$$A_{m,tt}^{(3)} + 2cA_{m,t}^{(3)} + \left(m^4+1\right)A_m^{(3)} = \frac{8\bar{a}_m m^6\, \mathrm{f}(\mathrm{t})\mathrm{F}(\mathrm{t})}{\bar{\xi}^2} + \frac{3A_m^{(1)3}}{4} \tag{3.17b}$$

$$A_m^{(3)}(0) = A_{m,t}^{(3)}(0) \tag{3.17c}$$

However, when $n = 3m$ in (3.17a), we get
$$A_{3m,tt}^{(3)} + 2cA_{3m,t}^{(3)} + \left(81m^4+1\right)A_{3m}^{(3)} = -\frac{A_m^{(1)\,3}}{4} \tag{3.17d}$$

$$A_{3m}^{(3)}(0) = A_{3m,t}^{(3)}(0) \tag{3.17c}$$

On solving (3.17b,c), we have

$$A_m^{(3)}(t) = \frac{8\bar{a}_m m^6}{\bar{\xi}^2}\int_0^t h(\tau)\,\mathrm{f}(\mathrm{t}-\tau)\,\mathrm{F}(\mathrm{t}-\tau)\,\mathrm{d}\tau + \frac{3}{4}\int_0^t \mathrm{h}(\tau)\,\mathrm{A}_\mathrm{m}^{(1)\,3}(\mathrm{t}-\tau)\,d\tau \tag{3.18a}$$

On further simplifying (3.18a) , we have

$$A_m^{(3)}(t) = \frac{8\bar{a}_m m^6}{\bar{\xi}^2}\int_0^t\int_0^{t-\tau_3}\int_0^{t-\tau_1-\tau_3} h(\tau_1)h(\tau_2)h(\tau_3)\, f(t\text{-}\tau_3)\, f(t\text{-}\tau_1-\tau_3)\, f(t\text{-}\tau_1-\tau_2-\tau_3)\, d\tau_1 d\tau_2 d\tau_3$$
$$+6\left(\bar{a}_m m^2\right)^3\int_0^t\int_0^{t-\tau_4}\int_0^{t-\tau_4}\int_0^{t\text{-}\tau_4} h(\tau_1)h(\tau_2)h(\tau_3)\, h(\tau_4)\, f(t\text{-}\tau_1-\tau_4)\, f(t\text{-}\tau_2-\tau_4)\, f(t\text{-}\tau_3-\tau_4) d\tau_1 d\tau_2 d\tau_3 d\tau_4 \tag{3.18b}$$

On solving (3.17d,e), we have

$$A_{3m}^{(3)}(t) = -\frac{1}{4}\int_0^t \tilde{\mathrm{h}}(\tau)A_m^{(1)3}(t-\tau)\,d\tau\ ,\ \tilde{\mathrm{h}}(\tau) = \frac{e^{-\mathrm{c}\tau}\sin\varphi_{3m}\tau}{\varphi_{3m}}\ ,\ \varphi_{3\mathrm{m}} = \sqrt{81m^4+1-c^2} \tag{3.19a}$$

On further simplifying (3.19a) ,we have

$$A_{3m}^{(3)}(t) = -2\left(\bar{a}_m m^2\right)^3\int_0^t\int_0^{t-\tau_4}\int_0^{t-\tau_4}\int_0^{t-\tau_4} h(\tau_1)h(\tau_2)h(\tau_3)\tilde{h}(\tau_4)\, f(t-\tau_1-\tau_4)\, f(t-\tau_2-\tau_4)\, f(t-\tau_3-\tau_4) d\tau_1 \Lambda\ d\tau_4 \tag{3.19b}$$

Hence, we write the displacement component $w^{(3)}$ and the overall displacement $w(x,t)$ as

$$w^{(3)} = A_m^{(3)}\sin\ mx + A_{3m}^{(3)}\sin\ 3mx, \tag{3.20}$$

$$w(x,t) = \in A_m^{(1)}\sin\ mx + \in^2 A_m^{(2)}\sin\ mx + \in^3\left(A_m^{(3)}\sin\ mx + A_{3m}^{(3)}\sin\ 3mx\right) + \Lambda \tag{3.21}$$

We note that the random load *f*(*t*) confers some element of randomness on the lateral displacement $w(x,t)$, whose maximum mean square ∇_a^2 we shall next determine subsequent upon which we [2,3] determine the dynamic buckling load λ_D from the maximization

$$\frac{d\lambda}{d\nabla_a^2} = 0 \tag{3.22}$$

3.1 Maximum mean square displacement ∇_a^2

We first determine the mean displacement $\left\langle w(x,t)\right\rangle$ which is necessary in the determination of the maximum displacement, thus:

$$\left\langle \mathrm{w(x,t)}\right\rangle = \in\left\langle \mathrm{A_m^{(1)}}\right\rangle \mathrm{sinmx} + \in^2\left\langle \mathrm{A_m^{(2)}}\right\rangle \mathrm{sinmx} + \in^3\left[\left\langle \mathrm{A_m^{(3)}}\right\rangle \mathrm{sinmx} + \left\langle \mathrm{A_{3m}^{(3)}}\right\rangle \mathrm{sin3mx}\right] + \Lambda \tag{3.23}$$

We shall now evaluate each of the terms in (3.23) where the upper limit of every integral hereafter is to be evaluated at infinity by virtue of the limiting process in equation (3.2b).Thus we have

$$\left\langle A_m^{(1)}\right\rangle = 2\bar{a}_m m^2\int_0^\infty h(\tau)\left\langle f(t-\tau)\right\rangle \mathrm{d}\tau = 0 \tag{3.24a}$$

where (3.24a) follows because *f*(*t*) is a zero-mean Gaussian function characterized by $\langle f(t) \rangle = 0$ .Similarly, we have

$$\langle A_m^{(2)} \rangle = \frac{4\bar{a}_m m^4}{\bar{\xi}} \int_0^\infty \int_0^\infty \mathrm{h}(\tau_1)\mathrm{h}(\tau_2) \langle f(t-\tau_1-\tau_2) f(t-\tau_1) \rangle d\tau_1 d\tau_2$$

$$= \frac{4\bar{a}_m m^4 R_0}{\bar{\xi}} \int_0^\infty \int_0^\infty \mathrm{h}(\tau_1)\mathrm{h}(\tau_2) \mathrm{R_f}(\tau_2) d\tau_1 d\tau_2 = \frac{2\bar{a}_m m^4 R_0 S_1}{\varphi_m(\varphi_m^2 + c^2)\bar{\xi}} \quad (3.24b)$$

$$S_1 = \left[\frac{\varphi_m + \Omega}{(\varphi_m + \Omega)^2 + (c+\alpha)^2} + \frac{\varphi_m - \Omega}{(\varphi_m - \Omega)^2 + (c+\alpha)^2} \right] \quad (3.24c)$$

Using (3.18b), we have

$$\langle A_m^{(2)} \rangle = \frac{8\bar{a}_m m^6}{\bar{\xi}} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)h(\tau_3) \langle f(t-\tau_3) f(t-\tau_1-\tau_3) f(t-\tau_1-\tau_2-\tau_3) \rangle d\tau_1 d\tau_2 d\tau_3$$

$$+(6\bar{a}_m m^2)^3 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4) \langle f(t-\tau_1-\tau_4) f(t-\tau_2-\tau_4) f(t-\tau_3-\tau_4) \rangle \times \ d\tau_1 \Lambda \ d\tau_4 \quad (3.25a)$$

The averaging process in each of the two three-point iterated integrals in (3.25a) easily takes the form $\langle f(t_1) f(t_2) f(t_3) \rangle$ which is simplified [10] in terms of a product of lower level correlations as

$$\langle f(t_1) f(t_2) f(t_3) \rangle = \langle f(t_1) f(t_2) \rangle \langle f(t_3) \rangle = 0 \quad (3.25b)$$

The result (3.25b) follows because $\langle f(t) \rangle = 0$ .We therefore conclude that

$$\langle A_m^{(3)} \rangle = 0 \quad (3.25c)$$

We now evaluate $\langle A_{3m}^{(3)} \rangle$, using (3.19a), as

$$\langle A_{3m}^{(3)} \rangle = -2(\bar{a}_m m^2)^3 \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{h}(\tau_1)\mathrm{h}(\tau_2)\mathrm{h}(\tau_3)\tilde{\mathrm{h}}(\tau_4) \langle f(t-\tau_1-\tau_4) f(t-\tau_2-\tau_4) f(t-\tau_3-\tau_4) \rangle \times d\tau_1 \Lambda \ d\tau_4 = 0 \quad (3.26)$$

The result (3.26) follows because the integrand is a three-point average, just as the case in (3.25a). We therefore conclude, using (3.25b,c) , that

$$\langle w(x,t) \rangle = \frac{2\bar{a}_m m^4 R_0 S_1}{\varphi_m(\varphi_m^2 + c^2)\bar{\xi}} \quad (3.27)$$

3.2 Mean square displacement, $\nabla^2(x,t)$

To determine $\nabla^2(x,t)$, we first determine the autocorrelation $R_w(x,t)$ of the displacement $w(x,t)$ in the following way

$$R_w(x_1, x_2, \tau) = \langle \ [\ \{\mathrm{w}(\mathrm{x}_1, t) - \langle w(x_1, t) \rangle \}\{\mathrm{w}(\mathrm{x}_2, t+\tau) - \langle w(x_2, t+\tau) \rangle \} \ ] \ \rangle \quad (3.28a)$$

The mean square displacement $\nabla^2(x,t)$ easily follows by setting $x_1 = x_2 = x$, $\tau = 0$, to yield

$$\nabla^2(x,t) = R_w(x,0) = \langle [\{ \mathrm{w(x,t)} - \langle \mathrm{w(x,t)} \rangle \}\{\mathrm{w(x,t)} - \langle \mathrm{w(x,t)} \rangle \}] \rangle = \langle (w(x,t))^2 \rangle - \langle w(x,t) \rangle^2 \quad (3.28b)$$

While , we can easily evaluate the term $\langle w(x,t) \rangle^2$ in (3.28b) by using equation (3.27), we shall however now determine in detail the term $\langle (w(x,t))^2 \rangle$ arising from (3.28b) by using (3.23). We note, using (3.23) that

$$\left\langle \left(w(x,t) \right)^2 \right\rangle = \Big\langle \Big[\in^2 A_m^{(1)2} \sin^2 mx + 2\in^3 A_m^{(1)} A_m^{(2)} \sin^2 mx + \in^4 \Big\{ A_m^{(2)2} \sin^2 mx + 2A_m^{(1)} A_m^{(3)} \sin^2 mx$$

$$+ 2A_m^{(1)} A_{3m}^{(3)} \sin mx \sin 3mx \Big\} + 2\in^5 \Big\{ A_m^{(2)} A_m^{(3)} \sin^2 mx + A_m^{(2)} A_m^{(3)} \sin mx \sin 3mx \Big\} + \in^6 \Big\{ A_m^{(3)2} \sin mx \quad (3.29)$$

$$+ A_m^{(3)} A_{3m}^{(3)} \sin mx \sin 3mx + A_{3m}^{(3)2} \sin^2 3mx \Big\} \Big] \Big\rangle + \Lambda$$

The averaging process in (3.29) is distributive over individual terms and so we shall now evaluate each term in (3.29), omitting ,tentatively, the spatial dependence. Thus we have the following for $\left\langle \left(A_m^{(1)} \right)^2 \right\rangle$,

$$\left\langle (A_m^{(1)})^2 \right\rangle = 4\bar{a}_m^2 m^4 \int_0^\infty \int_0^\infty h(\tau_1) h(\tau_2) \langle f(t-\tau_1) f(t-\tau_2) \rangle d\tau_1 d\tau_2 \quad (3.30a)$$

$$= 4\bar{a}_m^2 m^4 \int_0^\infty \int_0^\infty h(\tau_1) h(\tau_2) R_f(\tau_1 - \tau_2) d\tau_1 d\tau_2 = \frac{\bar{a}_m{}^2 m^4 T_1 R_0}{\varphi_m^2} \quad (3.30b)$$

where

$$T_1 = \left[\left\{ \frac{\varphi_m + \Omega}{(\varphi_m + \Omega)^2 + (c+\alpha)^2} + \frac{\varphi_m - \Omega}{(\varphi_m - \Omega)^2 + (c+\alpha)^2} \right\} \left\{ \frac{\varphi_m + \Omega}{(\varphi_m + \Omega)^2 + (c-\alpha)^2} + \frac{\varphi_m - \Omega}{(\varphi_m - \Omega)^2 + (c-\alpha)^2} \right\} \right] \quad (3.30c)$$

Similarly, for $\left\langle A_m^{(1)} A_m^{(2)} \right\rangle$,we have,

$$\left\langle A_m^{(1)} A_m^{(2)} \right\rangle = \frac{8\bar{a}_m^2 m^6}{\xi} \int_0^\infty \int_0^\infty \mathrm{h}(\tau_1) \mathrm{h}(\tau_2) \mathrm{h}(\tau_3) \langle \mathrm{f}(\mathrm{t}-\tau_1) \mathrm{f}(\mathrm{t}-\tau_2) \mathrm{f}(\mathrm{t}-\tau_2-\tau_3) \rangle d\tau_1 d\tau_2 d\tau_3 = 0 \quad (3.31)$$

We note that (3.31) follows from (3.25b). Similarly, for $\left\langle \left(A_m^{(2)} \right)^2 \right\rangle$,we have the following

$$\left\langle \left(A_m^{(2)} \right)^2 \right\rangle = \left(\frac{4\bar{a}_m m^4}{\xi} \right)^2 \underbrace{\int^\infty \Lambda \int^\infty}_{four\ \text{integrals}} h(\tau_1) \Lambda\, h(\tau_4) \langle f(t-\tau_1) f(t-\tau_1-\tau_2) f(t-\tau_3) f(t-\tau_3-\tau_4) \rangle d\tau_1 \Lambda\, d\tau_4 \quad (3.32a)$$

We note the following simplification for a four-point correlation that characterizes (3.32a)

$$\left\langle R_f(t_1) R_f(t_2) R_f(t_3) R_f(t_4) \right\rangle = R_f(t_2 - t_1) R_f(t_4 - t_3) + R_f(t_3 - t_1) R_f(t_4 - t_2) + R_f(t_4 - t_1) R_f(t_3 - t_2) \quad (3.32b)$$

We can therefore represent (3.32a) as a sum of products of three lower level correlations using (3.32b), as follows:

$$\left\langle \left(A_m^{(2)} \right)^2 \right\rangle = \left(\frac{4\bar{a}_m m^4}{\xi} \right)^2 \underbrace{\int^\infty \Lambda \int^\infty}_{four\ \text{integrtals}} h(\tau_1) \Lambda\, h(\tau_4) \big[R_f(\tau_2) R_f(\tau_4) + R_f(\tau_1 - \tau_3) R_f(\tau_2 - \tau_4)$$

$$+ R_f(\tau_1 - \tau_3 - \tau_4) R_f(\tau_1 + \tau_2 - \tau_3) \big] d\tau_1 \Lambda\, d\tau_4 \quad (3.32c)$$

By omitting, for now, the coefficient $\left(\frac{4\bar{a}_m m^4}{\xi} \right)^2$, we shall evaluate each of the three consecutive terms, here dubbed T_2 , T_3 and T_4, on the right hand side of (3.32c). Thus we have

$$T_2 = \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{four\ \text{integrals}} h(\tau_1)\Lambda\ h(\tau_4)R_f(\tau_2)R_f(\tau_4)d\tau_1\Lambda\ d\tau_4 = \left(\int_0^{\infty} h(\tau)d\tau\right)^2\left(\int_0^{\infty} h(\tau)R_f(\tau)d\tau\right)^2 = \frac{S_1^2R_0^2}{\left(\varphi_m^2+c^2\right)^2} \quad (3.32d)$$

From the second term in (3.32c) we have the following for T_3:

$$T_3 = \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{four\ \text{integrals}0} h(\tau_1)\Lambda\ h(\tau_4)R_f(\tau_1-\tau_3)R_f(\tau_2-\tau_4)d\tau_1\Lambda\ d\tau_4 = \left\{\int_0^{\infty}\int_0^{\infty} h(\tau_1)h(\tau_3)R_f(\tau_1-\tau_3)d\tau_1 d\tau_3\right\}^2 \quad (3.33a)$$

$$= \left\{\frac{R_0}{2\varphi_m}\int_0^{\infty} h(\tau_3)(S_1\cos\Omega\tau_3 + S_2\sin\Omega\tau_3)d\tau_3\right\}^2, S_2 = \left\{\frac{c+\alpha}{(\varphi_m+\Omega)^2+(c+\alpha)^2} + \frac{c+\alpha}{(\varphi_m-\Omega)^2+(c+\alpha)^2}\right\} \quad (3.33b)$$

$$T_3 = \frac{S_4R_0^2}{16\varphi_m^2}, S_4 = (S_1S_5+S_2S_6)^2, S_5 = \frac{(\varphi_m-\Omega)}{(\varphi_m-\Omega)^2+(c-\alpha)^2} + \frac{(\varphi_m+\Omega)}{(\varphi_m+\Omega)^2+(c-\alpha)^2} \quad (3.33c)$$

$$S_6 = \frac{c-\alpha}{(\varphi_m-\Omega)^2+(c-\alpha)^2} - \frac{c-\alpha}{(\varphi_m+\Omega)^2+(c-\alpha)^2} \quad (3.33d)$$

We now evaluate the term T_4 from the third term in (3.32c) as:

$$T_4 = \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{four\ \text{int}egrals} h(\tau_1)\Lambda\ h(\tau_4)R_f(\tau_1-\tau_3-\tau_4)R_f(\tau_1+\tau_2-\tau_3)d\tau_1\Lambda\ d\tau_4 \quad (3.34a)$$

Since $R_f(\tau)$ is an even function, we can rewrite (3.34a) as

$$T_4 = \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{four\text{int}egrals} h(\tau_1)\Lambda\ h(\tau_4)R_f(\tau_1-\tau_3-\tau_4)R_f(\tau_3-\tau_1-\tau_2)d\tau_1\Lambda\ d\tau_4 \quad (3.34b)$$

On evaluating (3.34b) with respect to τ_4, we have

$$T_4 = \frac{R_0}{2\varphi_m}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} h(\tau_1)h(\tau_2)h(\tau_3)R_f(\tau_3-\tau_1-\tau_2)e^{-\alpha(\tau_1-\tau_3)}\times[\,S_5\cos\Omega(\tau_1-\tau_3)+S_6\sin\Omega(\tau_1-\tau_3)\,]\,d\tau_1d\tau_2d\tau_3 \quad (3.34c)$$

On evaluating (3.34b) with respect to τ_2, we have

$$T_4 = \frac{R_0^2}{4\varphi_m^2}\int_0^{\infty}\int_0^{\infty} h(\tau_1)h(\tau_2)e^{-\alpha(\tau_3-\tau_1)}e^{-\alpha(\tau_1-\tau_3)}\,[\,\{S_5\cos\Omega(\tau_1-\tau_3)+S_6\sin\Omega(\tau_1-\tau_3)\}\times$$
$$\{S_5\cos\ \Omega(\tau_3-\tau_1)+S_6\sin\Omega(\tau_3-\tau_1)\ \}\ ]\,d\tau_1d\tau_3 \quad (3.34d)$$

If we simplify (3.34d) further, we get

$$T_4 = \frac{R_0^2}{8\varphi_m^2}\int_0^{\infty}\int_0^{\infty} h(\tau_1)h(\tau_3)(S_5^2-S_6^2)[1-\cos 2\Omega\tau_1\cos 2\Omega\tau_3 - \sin 2\Omega\tau_1\sin 2\Omega\tau_3]\,\mathrm{d}\tau_1 d\tau_3 \quad (3.34e)$$

On finally carrying out the two integrations in (3.34e), we have

$$T_4 = \frac{(S_5^2-S_6^2)S_7R_0^2}{8\varphi_m^2},\ S_7 = \left[\left(\frac{1}{\varphi_m^2+c^2}\right)^2 - \frac{1}{4\varphi_m^2}\left\{\left\{\left\{\frac{\varphi_m+2\Omega}{(\varphi_m+2\Omega)^2+c^2}+\frac{\varphi_m-2\Omega}{(\varphi_m-2\Omega)^2+c^2}\right\}\right.\right.\right.$$
$$\left.\left.\left.-\left\{\frac{c}{(\varphi_m-2\Omega)^2+c^2}-\frac{c}{(\varphi_m+2\Omega)^2+c^2}\right\}\right\}\right\}\right] \quad (3.35)$$

We thus conclude that

$$\left\langle \left(A_m^{(2)}\right)^2 \right\rangle = \left(\frac{4\bar{a}_m m^4 R_0}{\bar{\xi}}\right)^2 \left(T_2 + T_3 + T_4\right) \tag{3.36}$$

We shall now evaluate the term $\left\langle A_m^{(1)} A_m^{(3)} \right\rangle$ of order $\in^4$ in (3.29) thus :

$$\left\langle A_m^{(1)} A_m^{(3)} \right\rangle = \left[\frac{16\,\bar{a}_m^{\,2} m^8}{\bar{\xi}^{\,2}} \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{\textit{four}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\left\langle f(t-\tau_1)f(t-\tau_4)f(t-\tau_2-\tau_4)f(t-\tau_2-\tau_3-\tau_4)\right\rangle \right.$$

$$\times d\tau_1 \Lambda\, d\tau_4 + 12\left(\bar{a}_m m^2\right)^4 \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{\textit{five}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)\left\langle f(t-\tau_1)f(t-\tau_2-\tau_5)f(t-\tau_3-\tau_5)\times d\tau_1\Lambda\, d\tau_5 \tag{3.37a}$$

By omitting, for the time being, the coefficients outside the two integrals, here dubbed T_5 and T_6 respectively in (3.37a), we now simplify T_5 as:

$$T_5 = \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{\textit{four}\ \text{integrals}} h(\tau_1)\Lambda\ h(\tau_4)\left\langle \mathrm{f}(\mathrm{t}-\tau_1)f(\mathrm{t}-\tau_4)f(\mathrm{t}-\tau_2-\tau_4)f(\mathrm{t}-\tau_2-\tau_3-\tau_4)\right\rangle \mathrm{d}\tau_1\Lambda\ d\tau_4$$

$$= \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{\textit{four}\ \text{integrals}} h(\tau_1)\Lambda\ h(\tau_4)\left[\mathrm{R_f}(\tau_1-\tau_4)\mathrm{R_f}(-\tau_3)+\mathrm{R_f}(\tau_1-\tau_2-\tau_4)\mathrm{R_f}(-\tau_2-\tau_3)\right. \tag{3.37b}$$

$$+\mathrm{R_f}(\tau_1-\tau_2-\tau_3-\tau_4)\mathrm{R_f}(-\tau_{23})\left.\right]\mathrm{d}\tau_1\Lambda\ d\tau_4$$

The three terms in (3.37b) are here respectively denoted by T_{51} , T_{52} and T_{53} and we shall now evaluate each of them .Thus we have

$$T_{51} = \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{\textit{four}\text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)R_f(\tau_1-\tau_4)R_f(-\tau_3)\,\mathrm{d}\tau_1 d\tau_2 d\tau_3 d\tau_4 = \frac{R_0^2 S_1\left(S_5^2 + S_6 S_8\right)}{8_m^2\left(\varphi_m^2 + c^2\right)} \tag{3.38a}$$

$$S_8 = \left\{\frac{c+\alpha}{(\varphi_m-\Omega)^2+(c+\alpha)^2} - \frac{c+\alpha}{(\varphi_m+\Omega)^2+(c+\alpha)^2}\right\} \tag{3.38b}$$

We note that $R_f(\tau)$ is an even function. We also have, from the second term, namely, T_{52} , in (3.37b) as follows:

$$T_{52} = \underbrace{\int^{\infty}\Lambda\int^{\infty}}_{\textit{four}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\mathrm{R_f}(\tau_1-\tau_2-\tau_4)R_f(-\tau_2-\tau_3)\,\mathrm{d}\tau_1 d\tau_2 d\tau_3 d\tau_4 \tag{3.39a}$$

If we integrate (3.39a) with respect to τ_3 , we have

$$T_{52} = R_0 \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mathrm{h}(\tau_1)\mathrm{h}(\tau_2)\mathrm{h}(\tau_4)\mathrm{R_f}(\tau_1-\tau_2-\tau_4)\mathrm{e}^{-\alpha\tau_2}\left(S_1\cos\Omega\tau_2 + S_9\sin\Omega\tau_2\right)d\tau_1 d\tau_2 d\tau_4 \tag{3.39b}$$

$$S_9 = \left[\frac{c+\alpha}{(\varphi_m+\Omega)^2+(c+\alpha)^2} - \frac{c+\alpha}{(\varphi_m-\Omega)^2+(c+\alpha)^2}\right] \tag{3.39c}$$

On evaluating (3.39c) with respect to τ_1 , we get

$$T_{52} = R_0^2 \int_0^\infty\int_0^\infty \frac{e^{\alpha\tau_4}}{4\varphi_m^2} h(\tau_2)h(\tau_4)\{S_1\cos\Omega\tau_2 + S_9\cos\Omega\tau_2\}\times \{S_9\cos\Omega(\tau_2+\tau_4) + S_9\cos\Omega(\tau_2+\tau_4)\}d\tau_2 d\tau_4 \quad (3.39d)$$

On integrating (3.39d) with respect to τ_2, we have

$$T_{52} = \frac{R_0^2}{8\varphi_m^3}\int_0^\infty h(\tau_4)e^{\alpha\tau_4}\{S_{10}\cos\Omega\tau_4 + S_{11}\sin\Omega\tau_4\}d\tau_4 \quad (3.40a)$$

$$S_{10} = \left[\frac{S_1^2\varphi_m}{\varphi_m^2+\alpha^2} + S_1^2\left\{\frac{\varphi_m+\Omega}{(\varphi_m+\Omega)^2+\alpha^2} + \frac{\varphi_m-\Omega}{(\varphi_m-\Omega)^2+\alpha^2}\right\} + \frac{\alpha S_1S_9}{2}\left\{\frac{1}{(\varphi_m+2\Omega)^2+\alpha^2} + \frac{1}{(\varphi_m-2\Omega)^2+\alpha^2}\right\} + S_1S_9\alpha\left\{\frac{1}{(\varphi_m-2\Omega)^2+\alpha^2} - \frac{1}{(\varphi_m+2\Omega)^2+\alpha^2}\right\} + \frac{S_1S_9\varphi_m}{\varphi_m^2+\alpha^2}\right] \quad (3.40b)$$

$$S_{11} = \left[-\frac{S_1^2\alpha}{2}\left\{\frac{1}{(\varphi_m-2\Omega)^2+\alpha^2} - \frac{1}{(\varphi_m+2\Omega)^2+\alpha^2}\right\} + \frac{S_1S_9}{2}\left\{\frac{\varphi_m+2\Omega}{(\varphi_m+2\Omega)^2+\alpha^2} + \frac{\varphi_m-2\Omega}{(\varphi_m-2\Omega)^2+\alpha^2}\right\} + \frac{S_1S_9\alpha}{2}\left\{\frac{1}{(\varphi_m-2\Omega)^2+\alpha^2} + \frac{1}{(\varphi_m+2\Omega)^2+\alpha^2}\right\}\right] \quad (3.40c)$$

Thus, the final evaluation of (3.40a) gives

$$T_{52} = \frac{R_0^2}{16\varphi_m^4}(S_{10}S_5 + S_{11}S_6) \quad (3.41)$$

The term T_{53} of T_5 in (3.37b) is evaluated as follows:

$$T_{53} = \underbrace{\int_0^\infty \Lambda \int_0^\infty}_{\textit{four}\ \text{integrals}} h(\tau_1)\Lambda\, h(\tau_4) R_f(\tau_1-\tau_2-\tau_3) R_f(-\tau_2) d\tau_1 \Lambda\, d\tau_4 \quad (3.42a)$$

If we evaluate (3.42a) with respect to τ_1, we get

$$T_{53} = \frac{R_0}{2\varphi_m}\int_0^\infty\int_0^\infty\int_0^\infty h(\tau_2)h(\tau_3)h(\tau_4)R_f(-\tau_2)\, e^{\alpha(\tau_2+\tau_3+\tau_4)} \times [S_1\cos\Omega(\tau_2+\tau_3+\tau_4) - S_8\sin\Omega(\tau_2+\tau_3+\tau_4)]\, d\tau_2 d\tau_3 d\tau_4 \quad (3.42b)$$

On integrating (3.42b) with respect to τ_3, we have

$$T_{53} = \frac{R_0^2}{4\psi_m^2}\int_0^\infty\int_0^\infty h(\tau_2)h(\tau_4)\left[(S_1^2 - S_6S_8)\cos\Omega(\tau_2+\tau_4) - \{S_1S_6 + S_5S_8\sin\Omega(\tau_2+\tau_4)\}\right]d\tau_2 d\tau_4 \quad (3.42c)$$

On evaluating (3.42c) with respect to τ_4, we get

$$T_{53} = \frac{R_0^2}{8\varphi_m^3}\int_0^\infty h(\tau_2)e^{\alpha\tau_2}R_f(-\tau_2)[S_{12}\cos\Omega\tau_2 + S_{14}\sin\Omega\tau_2]d\tau_2 \quad (3.43a)$$

$$S_{12} = [S_5(S_1^2 - S_6S_8) - S_6(S_1S_6 + S_5S_8)],\ S_{13} = -[S_5(S_1^2 - S_6S_8) + S_6(S_1S_6 + S_5S_8)] \quad (3.43b)$$

On finally evaluating (3.43b), we get

$$T_{53} = \frac{R_0^2}{16\varphi_m^4}(S_{12}S_5 + S_6S_{13}) \quad (3.43c)$$

It thus follows from (3.37a,b), (3.38b), (3.41) and (3.43c) that

$$T_5 = \left(\frac{16\bar{a}_m^2 m^8}{\bar{\xi}^2}\right) \mathrm{R}_0^2 \left[\frac{S_1\left(S_5^2 + S_6 S_8\right)}{8\left(\varphi_m^2 + c^2\right)\varphi_m^2} + \frac{\left(S_{10}S_5 + S_{11}S_6\right)}{16\varphi_m^4} + \frac{\left(S_{12}S_5 + S_6 S_{13}\right)}{16\varphi_m^4}\right] \qquad (3.44)$$

We now evaluate the second term , T_6 of (3.37a) thus :

$$T_6 = 12\left(\bar{a}_m m^2\right)^4 \underbrace{\int_1^\infty \Lambda \int_3^\infty}_{five\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)\langle f(t-\tau_1) f(t-\tau_2-\tau_5) f(t-\tau_3-\tau_5)\times \qquad (3.45a)$$
$$f(t-\tau_4-\tau_5)\rangle \ ]d\tau_1 \Lambda\ d\tau_5$$

$$= 12\left(\bar{a}_m m^2\right)^4 \underbrace{\int_1^\infty \Lambda \int_3^\infty}_{five\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)\left[\mathrm{R_f}(\tau_1-\tau_2-\tau_5)\mathrm{R_f}(\tau_3-\tau_4)\right. \qquad (3.45b)$$
$$\left.+ \mathrm{R_f}(\tau_1-\tau_3-\tau_5)\mathrm{R_f}(\tau_2-\tau_4) + \mathrm{R_f}(\tau_1-\tau_4-\tau_5)\mathrm{R_f}(\tau_2-\tau_3)\ \right]\mathrm{d}\tau_1 \Lambda\ d\tau_5$$

The values of the three integrals in (3.45b) are equal on evaluation and so we shall evaluate only the first one , here dubbed T_{61}, thus :

$$T_{61} = \underbrace{\int_1^\infty \Lambda \int_3^\infty}_{five\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)\,\mathrm{R_f}(\tau_1-\tau_2-\tau_5)\,\mathrm{R_f}(\tau_3-\tau_4)d\tau_1\Lambda\ d\tau_5 \qquad (3.45c)$$

On evaluating (3.45c) first with respect to τ_1, and next, with respect to τ_3, we have

$$T_{61} = \frac{R_0^2}{4\varphi_m^2}\int_0^\infty\int_0^\infty\int_0^\infty h(\tau_2)h(\tau_4)h(\tau_5)e^{\alpha\tau_4} e^{\alpha(\tau_2+\tau_5)}\left[\ \{\mathrm{S}_1\cos\Omega\tau_4 + \mathrm{S}_8\sin\Omega\tau_4\}\times\right. \qquad (3.45d)$$
$$\left.\{\mathrm{S}_1\cos\Omega(\tau_2+\tau_5) + \mathrm{S}_8\sin\Omega(\tau_2+\tau_5)\ \}\ \right]d\tau_2 d\tau_4 d\tau_5$$

On evaluating (3.45d) with respect to τ_5, we have

$$T_{61} = \frac{R_0^2}{8\varphi_m^3}\int_0^\infty\int_0^\infty \mathrm{h}(\tau_2)\mathrm{h}(\tau_4)\mathrm{e}^{\alpha(\tau_2+\tau_4)}\left[\{\mathrm{S}_1\cos\Omega\tau_4 + \mathrm{S}_8\sin\ \Omega\tau_4\}\{(\mathrm{S}_1\mathrm{S}_5 + S_6S_8)\cos\Omega\tau_2\right. \qquad (3.45e)$$
$$\left. + (\mathrm{S}_5\mathrm{S}_8 - S_1S_6)\sin\Omega\tau_2\ \}\right]\mathrm{d}\tau_2 d\tau_4$$

On evaluating (3.45e), first with respect to τ_2 and lastly with respect to τ_4, we have

$$T_{61} = \frac{R_0^2}{32\varphi_m^5}\left(S_1S_5 + S_6S_8\right)\left[S_5\left(S_1S_5 + S_6S_8\right) + S_6\left(S_5S_8 - S_1S_6\right)\right] \qquad (3.45f)$$

Thus, from (3.45a-f), we have

$$T_6 = \frac{9\left(\bar{\mathrm{a}}_\mathrm{m} m^2\right)^4 \mathrm{R}_0^2}{8\,\varphi_m^5}\left(S_1S_5 + S_6S_8\right)\left[S_5\left(S_1S_5 + S_6S_8\right) + S_6\left(S_5S_8 - S_1S_6\right)\right] \qquad (3.45g)$$

It follows from (3.37a,b) that $\qquad \left\langle A_m^{(1)}A_m^{(3)}\right\rangle = T_5 + T_6 \qquad (3.45h)$

We next evaluate $\left\langle A_m^{(1)}A_{3m}^{(3)}\right\rangle$ as

$$\left\langle A_m^{(1)} A_{3m}^{(3)} \right\rangle = -4\left(\bar{a}_m m^2\right)^4 \underbrace{\int_{1}^{\infty} \Lambda \int_{3}^{\infty}}_{\textit{five}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\tilde{h}(\tau_5) \langle f(t-\tau_1) f(t-\tau_2-\tau_5) f(t-\tau_3-\tau_5) \times$$

$$f(t-\tau_4-\tau_5) \rangle \,]\, d\tau_1 \Lambda\ d\tau_5 \tag{3.46a}$$

$$= -4\left(\bar{a}_m m^2\right)^4 \underbrace{\int_{1}^{\infty} \Lambda \int_{3}^{\infty}}_{\textit{five}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\tilde{h}(\tau_5) [\, \mathrm{R_f}(\tau_1-\tau_2-\tau_5)\mathrm{R_f}(\tau_3-\tau_4) \times$$

$$+\mathrm{R_f}(\tau_1-\tau_3-\tau_5)\mathrm{R_f}(\tau_2-\tau_4)+\mathrm{R_f}(\tau_1-\tau_4-\tau_5)\mathrm{R_f}(\tau_2-\tau_3)\ ]\, \mathrm{d}\tau_1 \Lambda\ d\tau_5 \tag{3.46b}$$

All the three integrals in (3.46b) are equal on evaluation and so we shall evaluate only the first one, here dubbed T_{71} , omitting the coefficients of the integrals at the moment. By integrating (3.46b) first, with respect to τ_1 and next, with respect to τ_3 , we have

$$T_{71} = \frac{R_0^2}{4\varphi_m^2} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \mathrm{h}(\tau_2)\,\mathrm{h}(\tau_4)\,\tilde{\mathrm{h}}(\tau_5) [\, \{\mathrm{S_1} \cos \Omega(\tau_2+\tau_5) + \mathrm{S_8} \sin \Omega(\tau_2+\tau_5)\ \} \times$$

$$\{\mathrm{S_1} \cos \Omega \tau_4 + S_8 \sin \Omega \tau_4\ \}\ ]\, \mathrm{d}\tau_2 d\tau_4 d\tau_5 \tag{3.46c}$$

On evaluating (3.46c) with respect to τ_2 , we have

$$T_{71} = \frac{R_0^2}{8\varphi_{\mathrm{m}}^3} \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{\,\alpha(\tau_4+\tau_5)} h(\tau_4) h(\tau_5)\ [\ \{ (\mathrm{S_1}S_5 - S_8 S_6)\cos\Omega\tau_5 + (\mathrm{S_5}S_8 - S_1 S_6)\sin\Omega\tau_5 \} \times$$

$$\{\mathrm{S_1}\cos\Omega\tau_4 + S_8 \sin\Omega\tau_4\}\,]\,\mathrm{d}\tau_4 \mathrm{d}\tau_5 \tag{3.46d}$$

On integrating (3.46d) first with respect to τ_4 and next, with τ_5 , we have

$$T_{71} = \frac{R_0^2 (S_1 S_5 + S_6 S_8)}{32\varphi_m^4 \varphi_{3m}} [\ S_{14}(S_1 S_5 - S_6 S_8) + S_{15}(S_5 S_8 - S_1 S_6)\ \ ] \tag{3.46e}$$

where

$$S_{14} = \left[\frac{\varphi_{3m} + \Omega}{(\varphi_{3m}+\Omega)^2 + (c-\alpha)^2} + \frac{\varphi_{3m} - \Omega}{(\varphi_{3m}-\Omega)^2 + (c-\alpha)^2} \right] \tag{3.46f}$$

$$\mathrm{S}_{15} = \left[\frac{(c-\alpha)}{(\varphi_{3m}-\Omega)^2 + (c-\alpha)^2} - \frac{(c-\alpha)}{(\varphi_{3m}+\Omega)^2 + (c-\alpha)^2} \right] \tag{3.46g}$$

Thus we have

$$\left\langle A_m^{(1)} A_{3m}^{(3)} \right\rangle = -12\left(\bar{a}_m m^2\right)^4\ T_{71} \tag{3.47}$$

We shall now evaluate terms of order $\in^5$ in (3.29). The first of such two terms, namely $\left\langle A_m^{(2)} A_m^{(3)} \right\rangle$, (omitting spatial dependence for now), takes form

$$\left\langle A_m^{(2)}A_m^{(3)}\right\rangle = \frac{32\bar{a}_m^2 m^{10}}{\bar{\xi}^3}\underbrace{\int_1^\infty \Lambda \int_3^\infty}_{\textit{five}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)\langle f(t-\tau_1)f(t-\tau_1-\tau_2)f(t-\tau_5)\times$$

$$f(t-\tau_3-\tau_5)f(t-\tau_3-\tau_4-\tau_5)\rangle\ ]d\tau_1\Lambda\ d\tau_5 \tag{3.48}$$

$$+\frac{864\bar{a}_m^4 m^{10}}{\bar{\xi}}\underbrace{\int_1^\infty \Lambda \int_3^\infty}_{\textit{six}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)h(\tau_6)\langle f(t-\tau_1)f(t-\tau_1-\tau_2)f(t-\tau_3-\tau_6)\times$$

$$f(t-\tau_4-\tau_6)f(t-\tau_5-\tau_6)\rangle\ ]d\tau_1\Lambda\ d\tau_6$$

Each of the two integrals in (3.48) is a five-point correlation average which is here approximated thus:

$$\langle f(t_1)f(t_2)f(t_3)f(t_4)f(t_5)\rangle \cong \langle f(t_1)f(t_2)f(t_3)\rangle\langle f(t_4)f(t_5)\rangle = 0 \tag{3.49}$$

The result in (3.49) follows on account of (3.25b).It thus follows that $\left\langle A_m^{(2)}A_m^{(3)}\right\rangle = 0$ (3.50a)

By the same reasoning, it similarly follows, after some simplifications, that

$$\left\langle A_m^{(2)}A_{3m}^{(3)}\right\rangle = 0 \tag{3.50b}$$

We now display a typical term , of order $\in^6$ in (3.29) and show why terms of this order are here neglected. The first term of this order is $\left\langle \left(A_m^{(3)}\right)^2\right\rangle$, where we have omitted the spatial dependence for now. Using (3.18b), we expand this term as

$$\left\langle \left(A_m^{(3)}\right)^2\right\rangle = \frac{64\bar{a}_m^2 m^{12}}{\bar{\xi}^4}\underbrace{\int_1^\infty \Lambda \int_3^\infty}_{\textit{six}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)h(\tau_6)\langle f(t-\tau_3)f(t-\tau_1-\tau_3)\times f(t-\tau_1-\tau_2-\tau_3)$$

$$f(t-\tau_4-\tau_6)f(t-\tau_3-\tau_5-\tau_6)\,\mathrm{d}\tau_1\Lambda\ \mathrm{d}\tau_6 + \frac{96\bar{a}_m^4 m^{12}}{\bar{\xi}^2}\underbrace{\int_1^\infty \Lambda \int_3^\infty}_{\textit{seven}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)(\tau_5)h(\tau_6)h(\tau_7)\times$$

$$\langle f(t-\tau_3)f(t-\tau_1-\tau_3)f(t-\tau_1-\tau_2-\tau_3)f(t-\tau_4-\tau_7)f(t-\tau_5-\tau_7)f(t-\tau_6-\tau_7)\rangle d\tau_1\Lambda\ d\tau_7$$

$$+36\left(\bar{a}_m m^2\right)^6\underbrace{\int_1^\infty \Lambda \int_3^\infty}_{\textit{eight}\ \text{integrals}} h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)(\tau_5)h(\tau_6)h(\tau_7)h(\tau_8)\langle f(t-\tau_1-\tau_4)f(t-\tau_2-\tau_4)\times \tag{3.51}$$

$$f(t-\tau_3-\tau_4)f(t-\tau_5-\tau_8)f(t-\tau_6-\tau_8)f(t-\tau_7-\tau_8)\rangle d\tau_1\Lambda\ d\tau_8$$

Each of the three iterated integrals in (3.51) is a six-point correlated average of the type $\langle f(t_1)f(t_2)f(t_3)f(t_4)f(t_5)f(t_6)\rangle, \mathrm{t}_1 \neq t_2 \neq \mathrm{t}_3 \neq t_4 \neq t_5 \neq t_6$ .In principle, each of the six point-point integrals can be expressed as a sum of fifteen lower level integrals with an enormous computational complexity . Here we shall however approximate each of six-point correlation in the three integrals in (3.51) in the following way:

$$\langle f(t_1)f(t_2)f(t_3)f(t_4)f(t_5)f(t_6)\rangle \cong \langle f(t_1)f(t_2)f(t_3)f(t_4)\rangle\langle f(t_5)f(t_6)\rangle \tag{3.52}$$

Thus, on final simplification, each of the three terms in (3.51), (and , in general , all the terms of order $\in^6$ in (3.29) will be multiplied by $\in^6 R_0^3$. Since we assume the range $0<\in<1$, $0<\mathrm{R}_0<1$, this order of multiplication is Mathematically insignificant compared to earlier terms, hence terms of order $\in^6$ are here omitted.

By collecting terms, we now write the mean square displacement $\nabla^2(x,t)$ as , (following (3.27), (3.28b) and (3.29))

$$\nabla^2(x,t) = R_0 \in^2 S_{16}\sin^2 \mathrm{mx} + \mathrm{R}_0^2 \in^4 \left[S_{17}\sin^2 mx + S_{18}\sin \mathrm{mx}\sin 3\mathrm{mx}\right] + \Lambda \quad (3.53a)$$

where

$$S_{16} = \frac{m^4 \bar{a}_m^2 m^2 T_1}{\varphi_m^2}, \quad S_{17} = \left[\left(\frac{4\bar{a}_m m^4}{\xi}\right)^2 \left\{\frac{S_1}{(\varphi_m^2 + c^2)} + \frac{S_4}{16\varphi_m^4} + \frac{S_7(S_5^2 - S_6^2)}{8\varphi_m^2}\right\}\right.$$

$$+ 2\left\{\left\{\frac{16\,\bar{a}_m^2 m^8}{\xi^2}\left\{\frac{S_1(S_5^2 + S_6 S_8)}{8\varphi_m^2(\varphi_m^2 + c^2)} + \frac{(S_5 S_{10} + S_6 S_{11})}{16\varphi_m^4} + \frac{(S_5 S_{12} + S_6 S_{13})}{16\varphi_m^4}\right\}\right.\right. \quad (3.53b)$$

$$\left.\left. + \frac{9(\bar{a}_m m^2)^4 (S_1 S_5 + S_6 S_8)\{(S_1 S_5 + S_6 S_8)S_5 + (S_5 S_8 - S_1 S_6)S_6\} + }{8\varphi_m^5}\right\}\right\}$$

$$\left. - \left(\frac{2\bar{a}_m m^4 S_1}{\varphi_m(\varphi_m^2 + c^2)\xi}\right)^2\right]$$

$$\mathrm{S}_{18} = -\frac{12(\bar{a}_m m^2)^4 T_7}{R_0^2} \quad (3.53c)$$

3.3 Dynamic buckling load λ_D

In order to use the maximization process (3.20b) ,we have to determine the maximum mean square displacement $\nabla_a^2 = \nabla^2(x_a, t_a)$, where x_a and t_a are the values of x and t respectively for such a maximum. We however observe that by virtue of the limiting process (3.2b), we have, abinitio, eliminated the dependence of $\nabla^2(x,t)$ on the time t. Thus, the only condition for the maximum $\nabla^2(x_a, t_a)$ is

$$\nabla^2,_x = 0 \quad (3.54a)$$

On substituting (3.53a) into (3.54a) and simplifying, we get $x_a = \frac{\pi}{2m}$ (3.54b)

where we have taken the least nontrivial value of x_a. Thus, on evaluating (3.53a-c) at $x_a = \frac{\pi}{2m}$, m = 1,2, 3,

.... we get

$$\nabla_a^2 = \in^2 C_1 + \in^4 C_2 + \Lambda \quad (3.55a)$$

$$C_1 = R_0 S_{16}, \ \mathrm{C}_2 = R_0^2(S_{17} - S_{18}) \quad (3.55b)$$

To carry out the maximization (3.22) , we [2,3] first reverse the series (3.55a,b) in the form

$$\in^2 = f_1(\nabla_a^2) + f_2(\nabla_a^2)^2 + \Lambda \quad (3.56a)$$

If we substitute for ∇_a^2 in (5.56a) from (5.55a,b) and equate the coefficients of $\in^2$ and $\in^4$, we have

$$f_1 = \frac{1}{C_1}, \mathrm{f}_2 = -\frac{C_2}{C_1^3} \quad (3.56b)$$

The maximization (3.22) is now easily executed through (3.56a), using (3.56b) to get , after some simplification

$$\nabla_a^2(\lambda_D) = -\frac{f_1}{2f_2} = \frac{C_1^2}{2C_2} \quad (3.56c)$$

where $\nabla_a^2(\lambda_D)$ is the value of ∇_a^2 at $\lambda = \lambda_D$. If we evaluate (3.56a) at $\lambda = \lambda_D$, using (3.56c) and taking cognizance of the fact that $\in = \lambda\bar{\xi}$ (which is now evaluated at $\lambda = \lambda_D$), we have

$$\in_D^2 = \frac{C_1}{4C_2} \tag{3.57}$$

where $\in_D$ is the value of $\in$ at $\lambda = \lambda_D$. On simplifying (3.57), we have

$$\lambda_D = \frac{1}{2\bar{\xi}R_0^{\frac{1}{2}}}\sqrt{\frac{S_{16}}{S_{17} - S_{18}}} \tag{3.58}$$

4.0 Analysis of result

The validity of equation (3.58) is guaranteed subject to the following limitations $0 < c < 1$, $0 < \alpha < 1$, $0 < \bar{\xi} < 1$ and $\varphi_m \neq \Omega$.. We clearly observe that if the mean square displacement is a suitable statistical characterization of the random displacement, then the dynamic buckling load is of order $R_0^{-\frac{1}{2}}$, where R_0 is the variance of the random load. Such a relationship is bound to be different depending on the statistical characterization used. Equation (60) gives a simple and straightforward formula for evaluating the dynamic buckling load λ_D.If we demand that the buckling mode be strictly in the shape of imperfection (11) then we have to disregard S_{18} and the result in this case is

$$\lambda_D = \frac{1}{2\bar{\xi}R_0^{\frac{1}{2}}}\sqrt{\frac{S_{16}}{S_{17}}} \tag{4.1}$$

We expect the value of λ_D from (4.1) to be less than that in (3.58).

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