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On a stochastically imperfect quadratic-cubic column pressurized by a random dynamic load

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Abstract

This investigation explores the dynamic stability of a randomly stochastic viscously damped finite imperfect column lying on a quadratic-cubic nonlinear elastic foundation but trapped by a dynamically random load applied just after the initial time. Non-vanishing statistical means and autocorrelations of both the imperfection and dynamic load are assumed. In particular, the autocorrelation of the imperfection is assumed to be correlated as an exponentially decaying function of the space variable. The statistical mean of the normal displacement is determined and is assumed to be a suitable parameter for determining the dynamic buckling load . The dynamic buckling load is determined asymptotically and various deductions are made. The result is particularized to that of a finite column on a purely cubic nonlinear elastic foundation.

1.0 Introduction

Stochasticity in both static and dynamical settings has long been investigated, particularly , as it pertains to the stability of elastic structures. Earlier investigations in this regard include Crandle [1,2] (and references there cited), and Caughey [3] (and references there cited), among others. Some of these investigations were directed at elastic model structures and not specifically at actual practical every-day engineering structures .In the terrain of elastic stability of structures, investigations on stochasticity have often been discussed in the context of random imperfections but seldom in the context of the loading or forcing function. The analysis presented here therefore has a two-fold departure from most of the existing literatures particularly ,in the sense that stochasticity , as it actually concerns practical engineering structures, is here discussed in a dynamical setting, and , besides, both the loading and imperfections are assumed random . Thus , while the loading is random with respect to the time variable, the imperfection ,on the other hand ,is random with respect to the space variable.

A finite column on a quadratic –cubic non-linear elastic foundation, is an imperfection-sensitive structure that was initially investigated by Hansen and Roorda [4,5] .Later, Elishakoff [6] discussed it in the context of reliability ,assuming a case of random imperfection while Ette [7] investigated the dynamic stability of the structure and assumed deterministic imperfection . The main substance of this study , therefore, is the determination of the dynamic buckling load of the structure using asymptotic approximations in a regular perturbation analysis in the case where both the loading history and imperfection are considered randomly stochastic and adorned with certain imbued non-vanishing statistical characterizations ,such as the statistical mean and autocorrelation .

2.0 Mathematical formulation

As in [6,7], the relevant differential equation satisfied by the normal displacement $W(X,T)$ of a viscously damped column (with linear viscous damping taken proportional to first degree of velocity) on a nonlinear quadratic-cubic elastic foundation,(see diagram), but trapped by a random dynamic load $P(T)$, where X and T are the spatial and time variables respectively, is

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$$m_0 W,_{TT} + c_0 W,_T + EIW,_{XXXX} + 2P(T)W,_{XX} + k_1 W - k_2 W^2 - \alpha k_3 W^3 = -2P(T)\frac{d^2\overline{W}}{dX^2}, \mathrm{T} > 0 \qquad (2.1)$$

$$W = W,_{XX} = 0 \text{ at } \mathrm{X} = 0, \pi;\ \mathrm{W}(\mathrm{X},0) = \mathrm{W},_{\mathrm{t}}(\mathrm{X},0) = 0,\ 0 < X < \pi \qquad (2.2)$$

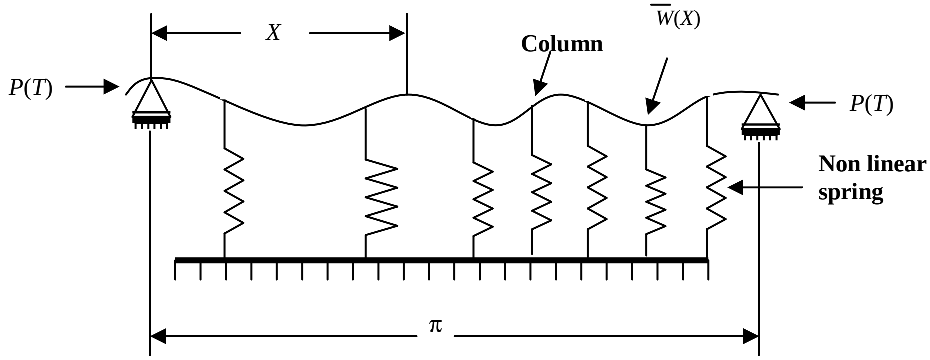

Figure 2.1: Column on a nonlinear elastic foundation

where a subscript following a comma indicates partial differentiation, m_0 is the mass per unit length, c_0 is the damping constant per unit length per velocity (assumed positive) and $k_1 > 0$, $k_2 > 0$ and $k_3 > 0$are positive constants while α is the imperfection-sensitivity parameter . For the special case $k_2 = 0$ (i.e. a column on a strictly cubic nonlinear elastic foundation), the resultant nonlinear foundation is said to be a "softening spring" if α =1, where as, it is said to be a "hardening spring" if α = -1. In equation (2.1), EI is the bending stiffness, where E and I are the Young's modulus and moment of inertia respectively. The nonlinear elastic foundation exerts a force per unit length of value $k_1W - k_2W^2 - \alpha k_3 W^3$ on the column. The time dependent load function $P(T)$ and twice differentiable stress-free imperfection $\overline{W}(X)$ are random functions, each, with a non-vanishing statistical mean and autocorrelation. We have neglected axial inertia as well as nonlinearities higher than the cubic .Also neglected are nonlinear derivatives of $W(X)$. We introduce the following nondimensional quantities

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X,\ \mathrm{w} = \left(\frac{\mathrm{k}_3}{\mathrm{k}_1}\right)^{\frac{1}{2}} W,\ \lambda\, \mathrm{f}(\mathrm{t}) = \frac{P(T)}{2(\mathrm{EL}\mathrm{k}_1)^{\frac{1}{2}}},\ \mathrm{t} = \left(\frac{\mathrm{k}_1}{\mathrm{m}_0}\right)^{\frac{1}{2}} T,\ \bar{\xi}\overline{\mathrm{w}} = \left(\frac{\mathrm{k}_3}{\mathrm{k}_1}\right)^{\frac{1}{2}} \overline{W} \qquad (2.3a)$$

$$\beta = \frac{k_2}{(k_1 k_3)^{\frac{1}{2}}},\ 2\mathrm{c} = \frac{\mathrm{c}_0}{(\mathrm{m}_0 k_1)^{\frac{1}{2}}},\quad 0 < c < 1, 0 < \lambda < 1, 0 < \bar{\xi} < 1 \qquad (2.3b)$$

On introducing (2.3a,b) into (2.1) and (2.2), we have

$$w,_{tt} + 2c\, w,_t + w,_{xxxx} + 2\lambda f(t)\, w,_{xx} + w - \beta\, w^2 - w^3 = -2\lambda \in f(t)\frac{d^2\overline{w}}{dx^2}, \mathrm{t} > 0 \qquad (2.4a)$$

$$\mathrm{w} = \mathrm{w},_{\mathrm{xx}} = 0 \text{ at } \mathrm{x} = 0, \pi;\ \mathrm{w}(\mathrm{x},0) = w,_t(x,0) = 0,\ 0 < \mathrm{x} < \pi \qquad (2.4b)$$

Here, λ is the nondimensional amplitude of the random load $f(t)$ while $\bar{\xi}$ is the amplitude of the random imperfection $\overline{w}(x)$.Analysis presented here was developed in [8] but the exposition here is strikingly different in context from that in (8). We assume the statistical mean $\langle \overline{w}(x) \rangle$ of imperfection $\overline{w}(x)$ to be given by

$$\langle \overline{w}(x) \rangle = r_1 ,\ 0 < r_1 < 1 \ ; \langle \Lambda \rangle = \frac{1}{\pi}\int_0^{\pi} (\Lambda)\, dx \text{ where } \langle \Lambda \rangle \qquad (2.5a)$$

is the Mathematical expectation. Similarly, the statistical mean, $E[f(t)]$, of the random load $f(t)$ is given by

$$E[f(t)] = r_0 ,\ 0 < r_0 < 1 ,\ E[\Lambda] = \lim T \to \infty \ \frac{1}{T}\int_0^{T} (\Lambda)\, dt \qquad (2.5b)$$

Elishakoff [6] specified some forms of the autocorrelation $R_{\overline{w}}(x_x, x_2)$ of the imperfection $\overline{w}(x)$ and these include the following:

$$(a) \quad R_{\overline{w}}(x_1, x_2) = D \exp\{-G|x_1 - x_2|\}, \quad (b) \quad R_{\overline{w}}(x_1, x_2) = \frac{2Q \sin T(x_1 - x_2)}{x_1 - x_2} \qquad (2.5c)$$

$$(c) \quad R_{\overline{w}}(x_1, x_2) = C_{PP} \sin(p\pi x_1) \sin(p\pi x_2) \qquad (2.5d)$$

where D, G,Q, T, P and C_{pp} are positive constants and x_1 and x_2 are two distinct points on the column. In our quest for the solution of (2.4a,b), we are to obtain a particular value of λ, called the dynamic buckling load , denoted by λ_D and is defined as the largest load parameter for which the solution of equations (2.4a,b) remains bounded for all time $t > 0$. According to Elishakoff [6,9], and Ette [8], the condition for obtaining λ_D is

$$\frac{d\lambda}{d\nabla_a} = 0 \qquad (2.6)$$

where ∇_a is the maximum mean displacement, assumed to be a suitable statistical characterization of the dynamic response statistic. It turns out that the mean displacement $\nabla(x,t)$,which we shall hereafter initiate its determination, actually depends on the statistical means and autocorrelations of both $f(t)$ and $\overline{w}(x)$.

3.0 Static analysis

Elishakoff [6, equation 30], in an *mth* term solution, using a slightly different non-dimensionlization process for the case of deterministic imperfection, obtained the static buckling load α^* in the following form

$$\left(\alpha_m - \alpha^* + \frac{s_1^2}{3\gamma_1}\right)^3 = \frac{81}{32}\left(\frac{m_*}{m}\right)^2 s_2\left[-(\alpha_m - \alpha^*)\frac{s_1}{3\gamma_1} - \frac{3_1^3}{27\gamma_1^2} - \alpha^*\overline{\xi}_m\right] \qquad (3.1a)$$

$$\gamma_1 = \frac{2}{3}\left(\frac{m_*}{m}\right)^2 s_1 \qquad (3.1b)$$

where all the terms in (3.1a,b) are as per the nomenclature used in [6].

3.1 Solution of the problem

We first obtain a detailed asymptotic derivation of normal displacement $w(x,t)$ subsequent upon which we determine $\nabla(x,t)$.We let $\in = \lambda\overline{\xi}$ and substitute same into (2.4a,b) to get

$$w,_{tt} + 2cw,_t + w,_{xxxx} + \frac{2\in f(t) w,_{xx}}{\overline{\xi}} + w - \beta w^2 - w^3 = -2\lambda\in f(t)\frac{d^2\overline{w}}{dx^2}, \ t > 0 \qquad (3.2a)$$

We let

$$w(x,t) = \sum_{i=1}^{\infty} w^{(i)}(x,t)\in^i \ , \quad \overline{w}(x) = \overline{a}_m \sin mx ,\ m = 1,2,3,\Lambda \qquad (3.2b)$$

and substitute (3,2b) into (3.2a) , equating coefficients of powers of $\in$ to get

$$Lw^{(1)} \equiv w^{(1)}_{,tt} + 2cw^{(1)}_{,t} + w^{(1)}_{,xxxx} + w = -2f(t)\frac{d^2\overline{w}}{dx^2} \qquad (3.3)$$

$$Lw^{(2)} = -2f(t)\frac{w^{(1)}_{,xx}}{\bar{\xi}} + \beta \mathrm{w}^{(1)\,2} \tag{3.4}$$

$$Lw^{(3)} = -2f(t)\frac{w^{(2)}_{,xx}}{\bar{\xi}} + 2\beta \mathrm{w}^{(1)}\mathrm{w}^{(2)} + \mathrm{w}^{(1)\,3} \tag{3.5}$$

$$w^{(\mathrm{i})} = w^{(i)}_{,x\mathrm{x}} = 0 \quad \text{at } \mathrm{x} = 0, \pi \;; \mathrm{w}^{(\mathrm{i})}(x,0) = w^{(\mathrm{i})}_{,\mathrm{t}}(x,0) = 0, \quad 0 < \mathrm{x} < \pi, i = 1,2,3,\Lambda \tag{3.6}$$

The solution of equations (3.3) - (3.6) is effected by assuming

$$w^{(i)}(x,t) = \sum_{n=1}^{\infty} A^{(\mathrm{i})}_{\mathrm{n}}(t)\sin \mathrm{nx} \tag{3.7}$$

On substituting (3.7) into (3.3), using $\overline{w}(x)$ as in (3.2b), we have

$$A^{(1)}_{\mathrm{m},\mathrm{tt}} + 2cA^{(1)}_{m,\mathrm{t}} + (m^4+1)\mathrm{A}^{(1)}_{\mathrm{m}} = 2\,\bar{\mathrm{a}}_{\mathrm{m}}m^2 f(t)\;;\; \mathrm{A}^{(1)}_{\mathrm{m}}(0) = A^{(1)}_{m,\mathrm{t}}(0) = 0 \tag{3.8}$$

We solve (3.8) to get

$$A^{(1)}_m(t) = 2\bar{a}_m m^2 \int_0^t \mathrm{h}(\tau) f(t-\tau)\,\mathrm{d}\tau, \quad \mathrm{h}(\tau) = \frac{e^{-c\tau}\sin\varphi_m\tau}{\varphi_m}, \; \varphi_{\mathrm{m}} = \sqrt{m^4+1-c^2} \tag{3.9}$$

It therefore follows that

$$w^{(1)}(x,t) = A^{(1)}_m(t)\sin \mathrm{mx} \tag{3.10}$$

We next substitute into (3.4) for $w^{(1)}(x,t)$ from (3.10) and get

$$Lw^{(2)} = \frac{2m^2A^{(1)}_m f(t)\sin \mathrm{mx}}{\bar{\xi}} + \frac{\beta A^{(1)\,2}_m(1-\cos 2mx)}{2}, \mathrm{w}^{(2)}(x,0) = w^{(2)}_{,\mathrm{t}}(x,0) = 0 \tag{3.11a}$$

Using (3.7), for $i = 2$, we substitute in (3.11a) to get

$$A^{(2)}_{\mathrm{m,tt}} + 2cA^{(2)}_{m,\mathrm{t}} + (m^4+1)\mathrm{A}^{(2)}_{\mathrm{m}} = \frac{2\bar{\mathrm{a}}_{\mathrm{m}}m^2A^{(1)}_m f(t)}{\bar{\xi}} + \frac{\beta \mathrm{A}^{(1)\,2}_{\mathrm{m}} f(t)\delta_{\mathrm{n},\,2\mathrm{m}\text{-}1}}{3\pi(2\mathrm{m}\text{-}1)};\; \mathrm{A}^{(2)}_{\mathrm{m}}(0) = A^{(2)}_{m,\mathrm{t}}(0) = 0 \tag{3.11b}$$

where $\delta_{\mathrm{n},\,2\mathrm{m}\text{-}1}$ is the Dirac delta function. On solving (3.11b), we have

$$A^{(2)}_m(t) = \frac{2m^2}{\bar{\xi}}\int_0^t h(\tau) f(t-\tau)\mathrm{A}^{(1)}_{\mathrm{m}}(t-\tau)\,\mathrm{d}\tau + \frac{\beta\,\delta_{\mathrm{n},\,2\mathrm{m}\text{-}1}}{3\pi(2m-1)}\int_0^t h(\tau)\mathrm{A}^{(1)\,2}_{\mathrm{m}}(t-\tau)\,\mathrm{d}\tau \tag{3.12a}$$

On further simplifying (3.12a), we have

$$A^{(2)}_m(t) = \frac{4\bar{a}_m m^4}{\bar{\xi}}\int_0^t\int_0^{(t-\tau_1)} h(\tau_1)h(\tau_2) f(t-\tau_1) f(t-\tau_1-\tau_2)\,\mathrm{d}\tau_1 d\tau_2 + \frac{4m^3\bar{a}^3_m\beta\delta_{n,\,2m-1}}{3\pi}\times$$
$$\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)} f(t-\tau_1-\tau_2)\, f(t-\tau_1-\tau_3)\,\mathrm{d}\tau_1 d\tau_2 d\tau_3 \tag{3.12b}$$

It follows that

$$w^{(2)}(x,t) = A^{(2)}_m(t)\sin \mathrm{mx} \tag{3.12c}$$

We now substitute into (3.5), using (3.10) and (3.12c), to get

$$Lw^{(3)} = \frac{2m^2A^{(2)}_m f(t)\sin \mathrm{mx}}{\bar{\xi}} + \beta A^{(1)}_m A^{(2)}_m(1-\cos 2mx) + \frac{\alpha\,\mathrm{A}^{(1)\,3}_{\mathrm{m}}(3\sin \mathrm{mx} - \sin 3\mathrm{mx})}{4} \tag{3.13a}$$

$$\mathrm{w}^{(3)}(x,0) = w^{(3)}_{,\mathrm{t}}(x,0) = 0 \tag{3.13b}$$

On substituting (3.7) into (3.13a,b), for $i = 3$, and simplifying, we get, for $n = m$

$$A_{m}^{(3)},_{tt}+2cA_{m}^{(3)},_{t}+\left(m^4+1\right)A_{m}^{(3)}=\frac{2m^2A_m^{(2)}(t)f(t)}{\bar{\xi}}+\frac{2\beta A_m^{(1)}(t)A_m^{(2)}(t)\delta_{n,2m-1}}{3\pi(2m-1)}+\frac{3\alpha A_m^{(1)\,3}(t)}{4} \quad (3.14a)$$

$$A_m^{(3)}(0)=A_{m,t}^{(3)}(0)=0 \quad (3.14b)$$

However, when $n = 3m$ in (3.13a,b), we have

$$A_{3m}^{(3)},_{tt}+2cA_{3m}^{(3)},_{t}+\left(81m^4+1\right)A_{3m}^{(3)}=-\frac{16\beta A_m^{(1)}(t)A_m^{(2)}(t)\delta_{n,2m-1}}{15(2m-1)}-\frac{\alpha A_m^{(1)\,3}(t)}{4} \quad (3.15a)$$

$$A_{3m}^{(3)}(0)=A_{3m,t}^{(3)}(0)=0 \quad (3.15b)$$

On solving (3.14a,b), we have

$$A_m^{(3)}(t)=\frac{2m^2}{\bar{\xi}}\int_0^t h(\tau)A_m^{(2)}(t-\tau)f(t-\tau)\,d\tau+\frac{2\beta\beta_{n,2m-1}}{3\pi(2m-1)}\int_0^t h(\tau)A_m^{(1)}(t-\tau)A_m^{(2)}(t-\tau)\,d\tau+\frac{3\alpha}{4}\int_0^t h(\tau)A_m^{(1)\,3}(t-\tau)\,d\tau \quad (3.16)$$

After simplifying (3.16), we have

$$A_m^{(3)}(t)=\frac{2m^2}{\bar{\xi}}\left[\frac{4\bar{a}_m m^4}{\bar{\xi}}\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_2)}h(\tau_1)h(\tau_2)h(\tau_3)f(t-\tau_1)f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3)\,d\tau_1d\tau_2d\tau_3\right.$$

$$+\frac{4\bar{a}_m^2m^4\beta\delta_{n,2m-1}}{3\pi(2m-1)}\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_2)}\int_0^{(t-\tau_1-\tau_3)}h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)f(t-\tau_1)f(t-\tau_1-\tau_2-\tau_3)\times$$

$$\left.f(t-\tau_1-\tau_2-\tau_4)\,d\tau_1d\tau_2d\tau_3d\tau_4\right]+\frac{2\beta\beta_{n,2m-1}}{3\pi(2m-1)}\left[\frac{8\bar{a}_m^2m^6}{\bar{\xi}}\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_3)}h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\times\right.$$

$$f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3)f(t-\tau_1-\tau_3-\tau_4)\,d\tau_1d\tau_2d\tau_3d\tau_4$$

$$+\frac{4\bar{a}_m^2m^3\beta\delta_{n,2m-1}}{3\pi}\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_3)}\int^{(t-\tau_1-\tau_3)}h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3-\tau_4)\times$$

$$\left.f(t-\tau_1-\tau_3-\tau_5)\,d\tau_1\Lambda\,d\tau_5\right]$$

$$+6\alpha\left(\bar{a}_m m^2\right)^3\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3)f(t-\tau_1-\tau_4)\,d\tau_1\Lambda\,d\tau_4 \quad (3.17)$$

We similarly solve (3.15a,b) and get

$$A_{3m}^{(3)}(t)=-\frac{16\beta\,\delta_{n,2m-1}}{15(2m-1)\pi}\int_0^t\tilde{h}(\tau)A_m^{(1)}(t-\tau)A_m^{(2)}(t-\tau)\,d\tau-\frac{\alpha}{4}\int_0^t\tilde{h}(\tau)A_m^{(1)\,3}(t-\tau)\,d\tau\;;\;\tilde{h}(\tau)=\left(81m^4+c^2-1\right)^{\frac{1}{2}} \quad (3.18)$$

On simplifying (3.18), we get

$$A_{3m}^{(3)}(t) = -\frac{16\,\beta\beta_{n,2m-1}}{15\,\pi(2m\text{-}1)}\Bigg[\frac{4\bar{a}_m m^4}{\bar{\xi}}\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_3)}\tilde{h}(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)f(t-\tau_1-\tau_2)\times$$

$$f(t-\tau_1-\tau_3)f(t-\tau_1-\tau_3-\tau_4)\,d\tau_1\Lambda\,d\tau_4 + \frac{4\beta\,m^3\bar{a}_m^2\,\delta_{n,2m-1}}{3\,\pi}\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1-\tau_3)}\int_0^{(t-\tau_1-\tau_3)}\tilde{h}(\tau_1)h(\tau_2)\times$$

$$h(\tau_3)h(\tau_4)h(\tau_5)f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3-\tau_4)f(t-\tau_1-\tau_3-\tau_5)\,d\tau_1\Lambda\,d\tau_5\;\Bigg]$$

$$-2\,\alpha(\bar{a}_m m^2)^3\int_0^t\int_0^{(t-\tau_1)}\int_0^{(t-\tau_1)}\int^{(t-\tau_1)}\tilde{h}(\tau_1)h(\tau_2)h(\tau\;)h(\tau_4)f(t-\tau_1-\tau_2)f(t-\tau_1-\tau_3)f(t-\tau_1-\tau_4)\,d\tau_1\Lambda\,d\tau_4 \quad (3.19)$$

We thus have
$$w^{(3)}(x,t) = A_m^{(3)}(t)\,\sin\,mx + A_{3m}^{(3)}(t)\,\sin\,3mx \qquad (3.20a)$$

so that the displacement $w(x,t)$ now becomes

$$w(x,t) = w^{(1)}\in + w^{(2)}\in^2 + w^{(3)}\in^3 + \Lambda = \in A_m^{(1)}\sin \text{mx} + \in^2 A_m^{(2)}\sin \text{mx}$$
$$+\in^3\left(A_m^{(3)}\,\text{si mx} + A_{3m}^{(3)}\sin 3\text{mx}\right) + \Lambda \qquad (3.20b)$$

3.2 Mean Displacement

We first determine the mean $\langle\bar{a}_m\rangle$ of the random parameter $\bar{a}_m$, as well as the mean squares $R_{\bar{w}}(0)$ and $R_f(0)$ of both the imperfection $\bar{w}(x)$ and dynamic load $f(t)$ respectively, where both $R_{\bar{w}}(0)$ and $R_f(0)$ are obtained from the respective autocorrelations given by $R_{\bar{w}}(x_1, x_2)$ and $R_f(t_1, t_2)$.From the second of (3.2b) ,we have

$$\langle\bar{w}(x)\rangle = r_1 = \langle\bar{a}_m \sin\;mx\rangle \qquad (3.21a)$$

If we multiply (3.21a) by sin mx and integrate from 0 to π, we get

$$\langle\bar{a}_m\rangle = \frac{4r_1}{2m-1}, \quad \text{m} = 1,2,3,\Lambda \qquad (3.21b)$$

Similarly, the autocorrelation, $R_{\bar{w}}(\zeta)$,of $\bar{w}(x)$,is

$$R_{\bar{w}}(\zeta) = \langle\,[\bar{w}(x) - \langle\bar{w}\rangle\,][\,\bar{w}(x+\zeta) - \langle\bar{w}(x+\zeta)\rangle\,]\rangle \qquad (3.21c)$$

Thus, $R_{\bar{w}}(0)$ is obtained by setting $\zeta = 0$ in (3.21c) to get

$$R_{\bar{w}}(0) = \langle\,(\bar{w}(x)\,)^2\rangle - r_1^2 = 2QT \qquad (3.21d)$$

where we have used the autocorrelation $R_{\bar{w}}(x_1, x_2) = \frac{2Q\sin T\,(x_1 - x_2)}{x_1 - x_2}$ as in (2.5c). On substituting into (3.21d) for $\bar{w}(x)$ and simplifying, we get

$$\langle\bar{a}_m^2\rangle = \frac{8r_1(r_1^2 + 2QT)}{(2m-1)\,\pi}, \; \text{m} = 1,2,3,\Lambda \qquad (3.21e)$$

We now determine the statistical mean displacement , $\nabla(x,t)$, where $\nabla(x,t) = \langle\,\langle w(x,t)\rangle\,\rangle$ and where

$$\langle\,\langle\Lambda\rangle\,\rangle = \langle\Lambda\rangle\,\text{E}[\Lambda\,] \qquad (3.22)$$

The averaging process in (3.22 results from the observation that virtually each of the terms in (3.20b) can be written as a product of a function of the time variable ,on one hand , and another ,as a function of spatial variable, on the other hand . Besides, the averaging processes in space and time variables are independent of each other and this makes (3.22) possible. We shall assume the autocorrelation $R_f(t_1, t_2)$ of $f(t)$ to be given by

$$R_f(\tau) = E[f(t)\mathrm{f}(\mathrm{t}+\tau)] = R_0 e^{-\alpha_1|\tau|}, 0 < \mathrm{R}_0 < 1,\ 0 < \alpha_1 < 1 \tag{3.23}$$

Thus ,when $\tau = 0$,we have the variance (or mean square) of $f(t)$ given by $R_f(0) = R_0$. Following the limiting process in (2.5b), we remark that the upper limit of any integration with respect to the time variable is henceforth evaluated at infinity. Now, the mean displacement, $\nabla(x,t)$, takes the form

$$\nabla(x,t) = \langle \langle w(x,t) \rangle \rangle = \langle \langle \in A_m^{(1)} \sin \mathrm{mx} + \in^2 A_m^{(2)} \sin \mathrm{mx} + \in^3 (A_m^{(3)} \sin \mathrm{mx} + A_{3m}^{(3)} \sin 3\mathrm{mx}) + \Lambda \rangle \rangle \tag{3.24a}$$

The operation in (3.24a) is distributive over each term and so we have

$$\nabla(x,t) = [\in \langle \langle A_m^{(1)} \rangle \rangle \sin mx + \in^2 \langle \langle A_m^{(2)} \rangle \rangle \sin mx + \in^3 \{ \langle \langle A_m^{(3)} \sin mx \rangle \rangle + \langle \langle A_{3m}^{(3)} \sin 3mx \rangle \rangle \}] + \Lambda \tag{3.24b}$$

We shall now evaluate individual terms in (3.24b) . Thus, using (3.9) and (3.22) in simplifying $\langle \langle A_m^{(1)} \rangle \rangle$, we have

$$\langle \langle A_m^{(1)} \rangle \rangle = 2m^2 \langle \bar{a}_m \rangle \int_0^\infty h(\tau) \mathrm{E}[\mathrm{f}\ (\mathrm{t} - \tau)]\, d\tau = 2m^2 r_0 T_0 \langle \bar{a}_m \rangle,\ \mathrm{T}_0 = \left(\frac{1}{\varphi_m^2 + c^2} \right) \tag{3.25}$$

Similarly on using (3.12b),we evaluate $\langle \langle A_m^{(2)} \rangle \rangle$ as

$$\langle \langle A_m^{(2)} \rangle \rangle = \frac{4m^4 \langle \bar{a}_m \rangle}{\bar{\xi}} \int_0^\infty \int_0^\infty h(\tau_1) h(\tau_2) \mathrm{E}[\mathrm{f}\ (\mathrm{t} - \tau_1) \mathrm{f}\ (\mathrm{t} - \tau_1 - \tau_2)]\, \mathrm{d}\tau_1 \mathrm{d}\tau_2$$
$$+ \frac{4m^3 \beta \langle \bar{a}_m^2 \rangle \delta_{\mathrm{n},2m-1}}{3\pi} \int_0^\infty \int_0^\infty \int_0^\infty h(\tau_1) h(\tau_2) h(\tau_3) \mathrm{E}[\mathrm{f}\ (\mathrm{t} - \tau_1 - \tau_2) \mathrm{f}\ (\mathrm{t} - \tau_1 - \tau_3)]\, \mathrm{d}\tau_1 d\tau_2 d\tau_3 \tag{3.26a}$$

We note the following simplifications:

$$E[\mathrm{f}(\mathrm{t} - \tau_1) \mathrm{f}(\mathrm{t} - \tau_1 - \tau_2)] = R_0 R_f(-\tau_2) = R_0 R_f(\tau_2) \tag{3.26b}$$

$$E[\mathrm{f}(\mathrm{t} - \tau_1 - \tau_2) \mathrm{f}(\mathrm{t} - \tau_1 - \tau_3)] = R_0 R_f(\tau_3) = R_0 R_f(\tau_3) \tag{3.27}$$

Thus on substituting (3.26b,c) into (3.26a) and simplifying, we get

$$\langle \langle A_m^{(2)} \rangle \rangle = \frac{4m^4 R_0 T_0 T_1 \langle \bar{a}_m \rangle}{\bar{\xi}} + \frac{4m^3 \beta \langle \bar{a}_m^2 \rangle \mathrm{T}_0 T_1 T_2 \delta_{\mathrm{n},2m-1}}{3\pi};\ \mathrm{T}_1 = \left(\frac{1}{\varphi_m^2 + (c+\alpha)^2} \right),$$
$$\mathrm{T}_2 = \left(\frac{1}{\varphi_m^2 + (c-\alpha)^2} \right) \tag{3.28}$$

We now evaluate $\langle \langle A_m^{(3)} \rangle \rangle$, using (3.16) as follows

$$\langle \langle A_m^{(3)} \rangle \rangle = \frac{2m^2\ r_0 R_0 T_0^2 T_1^2}{\bar{\xi}} \left[\frac{4m^4 \langle \bar{\mathrm{a}}_\mathrm{m} \rangle}{\bar{\xi}} + \frac{4m^4 \beta \langle \sigma_m^2 \rangle \delta_{n,2m-1}}{3\pi} \right]$$
$$+ \frac{2r_0 \beta T_0 T_1 T_2 R_0 \langle \bar{a}_m^2 \rangle}{9(2m-1)} \left[8m^6 + \frac{4\beta \delta_{n,2m-1}}{\pi} \right]$$
$$+ 6\alpha\, \mathrm{m}^6 r_0 R_0 T_0 T_1 T_2 \langle \bar{a}_m^2 \rangle \langle \bar{\mathrm{a}}_\mathrm{m} \rangle\ ;\ \mathrm{T}_2 = \frac{1}{\varphi_m^2 + (c-\alpha)^2} \tag{3.29}$$

We next evaluate $\left\langle\left\langle A_{3m}^{(3)}\right\rangle\right\rangle$ in the following way:

$$\left\langle\left\langle A_{3m}^{(3)}\right\rangle\right\rangle = -\frac{16\beta\,\delta_{m,2n-1}}{15(2m-1)\,\pi}\left[\frac{4m^4\langle\bar{a}_m\rangle}{\bar{\xi}}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \tilde{h}(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)E[f(t-\tau_1-\tau_2)\,f(t-\tau_1-\tau_2)\,]\times\right.$$

$$d\tau_1\Lambda\, d\tau_4 + \frac{4\beta\, m^3\langle\bar{a}_m^2\rangle\,\delta_{m,2n-1}}{3\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \tilde{h}(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)\,E[f(t-\tau_1-\tau_2)\,f(t-\tau_1-\tau_3-\tau_4)\times \quad (3.30a)$$

$$f(t-\tau_1-\tau_3-\tau_5)\,]\;d\tau_1\Lambda\, d\tau_5 - \;2\alpha\, m^6\langle\bar{a}_m^3\rangle\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \tilde{h}(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\,E[f(t-\tau_1-\tau_2)\times$$

$$f(t-\tau_1-\tau_3)\,f(t-\tau_1-\tau_4)\,]\,d\tau_1\Lambda\, d\tau_4$$

The simplification of (3.30a) is similar to that in (3.29a) and this yields

$$\left\langle\left\langle A_{3m}^{(3)}\right\rangle\right\rangle = -\frac{32m^2\beta\, R_0\, r_0\delta_{n,2m-1}}{15\,\pi(2\text{m -1})}\left[\frac{4m^4\langle\bar{a}_m\rangle\,\mathrm{T}_0T_1T_2T_3}{\bar{\xi}} + \frac{4m^3\beta\langle\bar{a}_m^2\rangle\,\delta_{n,2m-1}\,\mathrm{T}_0T_1T_2T_3}{3\pi}\right] \quad (3.30b)$$

$$-2\alpha\, m^6\langle\bar{a}_m\rangle\langle\bar{a}_m^2\rangle R_0r_0T_1T_2T_2 \;\;,\quad \mathrm{T}_3 = \left(\frac{1}{\varphi_{3m}^2 + c^2}\right)$$

Though we have actually evaluated all the terms as expressed in $\nabla(x,t)$ in (30b), we shall, for the enforcement of clarity, not substitute for them at the moment.

3.3 Maximum mean displacement

The mean displacement $\nabla(x,t)$ actually depends only on the spatial variable x because we have automatically eliminated the time dependence by the limiting process as in the second term of (2.5b). Thus , the condition for maximum mean displacement is $\nabla,_x(x_a) = 0$ .On substituting this into (3.24b),we have

$$x_a = \frac{\pi}{2\,m},\; m = 1,2,3,\Lambda \quad (3.31)$$

where x_a is the critical value of x at maximum mean displacement. On evaluating (3.24b) at x_a , we have

$$\nabla_a = \nabla(x_a) = \in\left\langle\left\langle A_m^{(1)}\right\rangle\right\rangle + \in^2\left\langle\left\langle A_m^{(2)}\right\rangle\right\rangle + \in^3\left[\left\langle\left\langle A_m^{(3)}\right\rangle\right\rangle - \left\langle\left\langle A_{3m}^{(3)}\right\rangle\right\rangle\right] + \Lambda \quad (3.32)$$

We now substitute all relevant terms into (3.32) and get

$$\nabla_a = \in C_1 + \in^2 C_2 + \in^3 C_3 + \Lambda \quad (3.33a)$$

$$C_1 = 2m^2r_0T_0\langle\bar{a}_m\rangle,\; \mathrm{C}_2 = 4R_0T_0T_1\left[\frac{\langle\bar{a}_m\rangle m^2}{\bar{\xi}} + \frac{m^3\beta\langle\bar{a}_m^2\rangle\,\delta_{\mathrm{n},2m-1}}{3\,\pi}\right] \quad (3.33b)$$

$$C_3 = \frac{8\,\mathrm{m}^7R_0r_0T_0T_1^2}{\bar{\xi}}\left[\frac{\langle\bar{a}_m\rangle}{\bar{\xi}} + \frac{\beta\,\delta_{\mathrm{n},2m-1}}{3\,\pi}\right] + \frac{2\,r_0\beta\,R_oT_0T_1T_2\langle\bar{a}_m^2\rangle\delta_{n,2m-1}}{3\,\pi(2\mathrm{m}-1)}\left[\frac{8\,m^6}{\bar{\xi}} + \frac{4\,\mathrm{m}^3\delta_{n,2m-1}}{3\,\pi}\right]$$

$$+\frac{32m^2R_0r_0T_0T_1T_2T_3}{15(2\mathrm{m}-1)}\left[\frac{4\,m^4\langle\bar{a}_m\rangle}{\bar{\xi}} + \frac{4\beta\,m^3\langle\bar{a}_m^2\rangle\,\delta_{\mathrm{n},2m-1}}{3\,\pi}\right] + 2\,\alpha\,\mathrm{m}^6R_0r_0T_0T_1^2\langle\bar{a}_m\rangle\langle\bar{a}_m^2\rangle \quad (3.33c)$$

The dynamic buckling load λ_D is determined using equation (2.6). According to Amazigo [12] and Ette [10] , this is accomplished by reversing the series (3.33a) so that we have

$$\in = d_1\nabla_a + d_2\nabla_a^2 + d_3\nabla_a^3 + \Lambda \quad (3.34a)$$

By substituting for ∇_a in (3.34a) from (3.33a) and equating the coefficients of $\in$, $\in^2$ and $\in^3$,we have the following respective values

$$d_1 = \frac{1}{C_1},\ \mathrm{d}_2 = -\frac{C_2}{C_1^3}\ ,\ \mathrm{d}_3 = \frac{2C_2^2 - C_1C_3}{C_1^5} \tag{3.34b}$$

The maximization (2.6) is easily accomplished via (3.34a,b) to yield

$$d_1 + 2d_2\nabla_{aD} + 3d_3\nabla_{aD}^2 = 0 \tag{3.35a}$$

where ∇_{aD} is the value of ∇_a at $\lambda = \lambda_D$ (i.e. at buckling) . We obtain ∇_{aD} from (3.35a) as

$$\nabla_{aD} = \frac{1}{3d_3}\left\{-d_2 \pm \left(d_2^2 - 3d_1d_3\right)^{\frac{1}{2}}\right\} \tag{3.35b}$$

By way of simplification of (3.35b), we get

$$\left(d_2^2 - 3d_1d_3\right)^{\frac{1}{2}} = \pm\sqrt{\frac{3C_3}{C_1^5}\left(1 - \frac{5C_2^2}{3C_1C_3}\right)} \tag{3.35c}$$

Taking the negative of the two signs from the above square root, we have

$$-d_2 - \left(d_2^2 - 3d_1d_3\right)^{\frac{1}{2}} = -\left(d_2^2 - 3d_1d_3\right)^{\frac{1}{2}}\left[1 + \frac{1}{\left(d_2^2 - 3d_1d_3\right)^{\frac{1}{2}}}\right] = -\sqrt{\frac{3C_3}{C_1^5}\left(1 - \frac{5C_2^2}{3C_1C_3}\right)}\left[1 - \frac{\mathrm{C}_2}{\sqrt{\frac{3C_3}{C_1}\left(1 - \frac{5C_2^2}{3C_1C_3}\right)}}\right] \tag{3.35d}$$

We acknowledge that we have already used, as in (3.34b), the simplification

$$d_3 = \frac{2C_2^2 - C_1C_3}{C_1^5} = -\frac{C_3}{C_1^4}\left(1 - \frac{2C_2^2}{C_1C_3}\right) \tag{3.35e}$$

Now , the actual evaluation of ∇_{aD} , from (3.35b), using (3.35d,e), is

$$\nabla_{aD} = \sqrt{\frac{C_1^3\left(1 - \frac{5C_2^2}{3C_1C_3}\right)}{3C_3}}\left[\frac{1 - \frac{C_2\sqrt{C_1}}{\sqrt{3C_3\left(1 - \frac{5C_2^2}{3C_1C_3}\right)}}}{1 - \frac{2C_2^2}{C_1C_3}}\right] \tag{3.35f}$$

To determine the dynamic buckling load λ_D, we evaluate (3.34a) at $\lambda = \lambda_D$ and get, after multiplying it by 3

$$3\in_D = \nabla_{aD}\left\{3\left(d_1 + d_2\nabla_{aD}\right) + 3\mathrm{d}_3\nabla_{aD}^2\right\} \tag{3.35g}$$

where $\in_D$ is the value of $\in$ at $\lambda = \lambda_D$. If we make $3d_3\nabla_{aD}^2$ the subject in (3.35*a*), we get

$$3d_3\nabla_{aD}^2 = -\left(d_1 + 2d_2\nabla_{aD}\right) \tag{3.35h}$$

On substituting (3.35*h*) into (3.35*g*) and simplifying, we get

$$3\in_D = 3\lambda_D\bar{\xi} = \nabla_{aD}\left(2d_1 + d_2\nabla_{aD}\right) = 2\nabla_{aD}\left(1 - \frac{C_2\nabla_{aD}}{2C_1^2}\right) \quad (3.36)$$

We note that (3.36) gives the expression for evaluating the dynamic buckling load λ_D which can easily be achieved by substituting for ∇_{aD} from (3.35*f*) and we have similarly substituted for d_1 and d_2 into (3.36) from (3.40b).

3.3 Analysis of result

The result (3.36) is asymptotic in nature and despite its apparent length, it is a simple result that determines λ_D. directly because all the terms on the right hand side are virtually known and already evaluated and independent of λ_D. Specific results, such as for the case $\beta = 0$, can easily be obtained by substituting $\beta = 0$ in the result (3.36) . Alternatively, the same result can easily be reworked by setting $C_2 = 0$ from the beginning of the analysis to give

$$\lambda_D\bar{\xi} = \frac{2}{3\sqrt{3}}\left(\frac{C_1}{C_3}\right)^{\frac{1}{2}} \quad (3.37)$$

This gives the equivalent result for a column on a cubic nonlinear elastic foundation under random dynamic loading. The result is valid for

$\left|\frac{C_2\sqrt{C_1}}{3C_3\left(1-\frac{5C_2^2}{3C_1C_3}\right)}\right| < 1$, $\left|\frac{5C_2^2}{3C_1C_3}\right| < 1$, and $2\left|\frac{C_2^2}{C_1C_3}\right| < 1$. An approximate form of ∇_{aD} is

$$\nabla_{aD} \cong \sqrt{\frac{C_1^3\left(1-\frac{5C_2^2}{3C_1C_3}\right)}{3C_3}} \quad (3.38)$$

which is valid for $\left|\frac{5C_2^3}{3C_1C_3}\right| < 1$. In all the results so obtained, possible combinations of the various parameters include the following: $Q = \frac{0.01}{\pi}$, $T = \pi$, $k_1 = \pi^4$, $k_2 = k_3 = 0.1k_1$ and $Q = \frac{0.01}{2\pi}$,

$T = \frac{\pi}{3}, k_1 = (2\pi)^4, k_2 = 0.4k_1, k_3 = 0.k_1$ among others.

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