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Oscillation in solutions of stochastic delay differential equations with real coefficients and several constant time lags

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Abstract

We study the role played by multiplicative noise perturbation in the oscillatory and non-oscillatory behaviour of the solution of first order linear scalar stochastic delay differential equation (SDDE)

$$dX(t) = \left[aX(t) + \sum_{i=1}^{n} b_i x(t - r_i) \right] dt + \mu X(t) dB(t),$$

$$t \geq 0 \; X(t) = \psi(t), \; t \in [-r, 0]1$$

We explain the interplay between the time lags and the multiplicative noise in the oscillatory behaviour of the solutions of the SDDE. It is proved that the presence of the multiplicative noise ensures that all solutions of the SDDE oscillate under negative feedback even if its corresponding deterministic equation has a non-oscillatory solution.

Keywords: Stochastic delay differential equation, oscillation, non-oscillation, noise perturbation.

1.0 Introduction:

In the application of stochastic differential equations, it is usually assumed that the system modeled is independent of the past states and hence determined only by the instantaneous position. However, as a result of the noise disturbances and fluctuations in the real World, we see that a good mathematical model must put into account the position of such system at some unit of time behind. Stochastic delay differential equations (SDDEs) and Deterministic delay differential equations (DDEs) are used to model systems that account for the fluctuation of the real World as well as undisturbed systems with dead times [1,2,3,4].

In the last few decades, attention has been focused on the study of solutions of DDEs which are oscillatory. For instance Agwo [1] studied the DDE with real coefficients of the form

$$x(t) + \sum_{i=1}^{n} P_i x(t - r_i) = 0, \; where \; P_i \in \Re, \quad r_i \in \Re^+ \tag{1.1}$$

and presented a set of necessary and sufficient conditions for the oscillation of (1.1) by extending a set of conditions given in Gyori and Ladas [13]. Later, Li [15] introduced a new technique to analyze the generalized characteristic equation to obtain some infinite integral conditions for oscillation of (1.1). For a survey on oscillatory and non-oscillatory results of solutions of first order linear delay differential equations, we refer to Wang et al.[18]

Although a large number of paper articles and research monographs have been written on oscillation and non-oscillation in deterministic delay differential equations, very little attention has been focused on the contribution of multiplicative noise perturbation (of the Ito type) to the oscillatory and non-oscillatory behaviour in solutions of stochastic delay differential equations. The concepts of oscillation and non-oscillation were introduced into stochastic processes by Appleby and Buckwar [3] when they studied the almost sure oscillatory properties of the linear SDDE

$$\left.\begin{aligned} dX(t) &= \left(aX(t) + bX(t - r(t))\right) + \sigma X(t)dB(t), && t \geq 0 \\ X(t) &= \psi(t), && t \in \left[-\bar{r}, 0\right] \end{aligned}\right\} \tag{1.2}$$

where r(t) is a variable time delay, σ is a positive number and $\{B(t)\}_{t \geq 0}$ is a one-dimensional Brownian motion. The authors showed that noise can induce oscillation in the solution of Equ. (1.2) whenever the feedback intensity $b < 0$.

In the present paper, we extend the formalism of Appleby and Buckwar [3] to guarantee the study of the effect of multiplicative noise on oscillatory behaviour of the more general SDDE with constant delays of the form

$$\left.\begin{aligned} \mathrm{dX}(t) &= \left[aX(t) + \sum_{i=1}^{n} b_i X(t - r_i)\right]dt + \mu X(t)dB(t) \\ X(t) &= \psi(t), \qquad t \in \left[-r, 0\right] \end{aligned}\right\} \tag{1.3}$$

where $0 \leq r_i \leq r$, for i = 1, 2, …, n, are constant time lags, a, $b_i \in \Re$, I = 1, 2,…, n, μ is a positive number which measures the average impact of the fast fluctuating internal noise and $\{B(t)\}_{t \geq 0}$ is a one –dimensional Brownian motion defined on a complete probability space (Ω, F, P) with filtration $\{F(t)\}_{t \geq 0}$ satisfying the usual conditions and the initial function $\psi \in C([-r, 0], \Re)$. By solution of the SDDE (1.3), we mean a stochastic process $\{X(t)\}_{t \geq 0}$ defined on a probability triple (Ω, F, P) and with continuous sample paths which satisfies Equation (1.3) as well as its initial function ψ. Since the theory of oscillation of stochastic delay differential equations is a natural extension of the theory of oscillation of deterministic delay differential equations, we will always compare the oscillatory results of the SDDE with those of the corresponding deterministic delay differential equation

$$x'(t) = ax(t) + \sum_{i=1}^{n} b_i x(t - r_i) \tag{1.4}$$

which satisfies the same initial function as equation. (1.3), where a, $b_i \neq 0$, $r_i > 0$. We use lower case letters for ease of notations.

By solution of the deterministic delay differential equation (1.4), we mean a function $x \in C([t^* - \rho, \infty), \Re)$ for some t^*, where $\rho = \max_{1 \leq i \leq n} \{r_i\}$ satisfies Equation (1.4) for all $t \geq t^*$.

The paper is organized in three sections. Section 1 contains the general introduction. In section 2, we discuss certain preliminary results as well as the technique used in our main proofs. In section 3, we present the main results.

2.0 Preliminaries:

Throughout this paper, we let *(Ω,F,P)* be a complete probability space with filtration $\{F(t)\}_{t \geq 0}$, which is a natural one, that is, a family of increasing sub-σ - algebras of *F* such that for $0 \leq s < t < \infty$, we have $F_s \subset F_t \subset F$, it is right continuous and each $\{F(t)\}_{t \geq 0}$ contains all P-null sets in ***F***. . By $\{B(t)\}_{t \geq 0}$, we mean a one-dimensional Brownian motion defined on the probability triple *(Ω,F,P)*. We denote by $C([t_0,\infty), \Re)$, the set of all functions from the interval $[t_0,\infty)$ to $\Re$, which are continuous for $t \geq t_0$.

***Definition* 2.1**

A solution x(t) of the deterministic delay differential equation (1.4) (equivalently of the SDDE (1.3)) is called *an equilibrium or zero solution if* $x(t) \equiv 0$ *whenever the initial function* $\psi \equiv 0$.

A solution x(t) of a DDE defined on the interval $[T_x, \infty)$ *and satisfies*

$$Sup\left\{\left|x(t)\right| : t \geq T\right\} > 0, \text{ for all } T > T_x$$

is called a regular or non-trivial solution, that is $\left|x(t)\right| \neq 0$ *in any infinite interval* $[T_x, \infty)$.

A non-trivial solution x(t) of a DDE is said to be eventually or almost certainly positive if there exists $t_1 > 0$ *such that* $x(t) > 0$, *for all* $t \geq t_1$.

A non-trivial solution x(t) of a DDE is said to be eventually or almost certainly negative if there exists $t_1 > 0$ *such that* $x(t) < 0$, *for all* $t \geq t_1$.

***Definition* 2.2**

As it is customary for the deterministic delay differential equation (1.4), *a non-trivial solution x(t) is said to be oscillatory if it has arbitrarily large zeros. That is, for* $t \geq t_0$, *there exists a sequence* $\{t_n: x(t_n) = 0\}$ *of x(t) such that* $\lim_{n\to\infty} t_n = +\infty$. *Otherwise x(t) is said to be non-oscillatory.*

In 2005, Appleby and Buckwar [3] introduced this definition into stochastic processes as follows:

***Definition* 2.3**

A non-trivial continuous function $f:[0, \infty) \to \Re$ *is said to be oscillatory if the set* $W_f = \{t \geq t_0: f(t) = 0\}$ *satisfies* $\sup W_f = \infty$. *If a function is not oscillatory, it is said to be non-oscillatory*. This notion was extended to stochastic processes in the following intuitive manner:

A stochastic process $\{X(t,w)\}_{t \geq 0}$ defined on a probability space (Ω, F, P) and with continuous sample paths is said to be almost surely (a.s.) oscillatory if there exists $\Omega^* \subseteq \Omega$ with $P[\Omega^*] = 1$ such that for all $w \in \Omega^*$, the path $X(., w)$ is oscillatory, otherwise it is said to be non-oscillatory. Hence a stochastic process $\{X(t,w)\}_{t \geq 0}$ defined on a probability space (Ω, F, P) and with continuous sample paths is said to be almost surely (a.s.) non-oscillatory if there exists $\Omega^* \subseteq \Omega$ with $P[\Omega^*] = 1$ such that for all $w \in \Omega^*$, the path $X(., w)$ is non-oscillatory.

For us to establish the existence of oscillatory solutions of the SDDE (1.3), we first associate the solution X of the SDDE with the solution of a scalar linear non-autonomous delay differential equation

$$Z'(t) = -\sum_{i=1}^{n} P_i(t) Z(t - r_i) \tag{2.1}$$

Where r_i are constant delays and $P_i(.) \geq 0$ are random non –negative continuous functions defined on some almost sure set $\Omega^* \subset \Omega$ by

$$P_i(t,w) = \begin{cases} -be^{-\lambda r_i}\, e^{(-\mu(B(t)(w) - B(t-r_i)(w)))}, & for\ t > \underline{t} \\ -be^{-\lambda t - \mu B(t)(w)}, & for\, t \leq \underline{t} \end{cases} \tag{2.2}$$

$$where\ \lambda = \left(a - \frac{\mu^2}{2}\right),\ \underline{t} = \inf\{t > 0 : t - r_i = 0\}\ such\ that\ \ for\ all\ t > \underline{t},$$

$$t - r_i \geq 0\ and\ w \in \Omega$$

Here, $\boldsymbol{P_i}$ depend upon the increments of a standard Brownian motion $\{B(t)\}_{t \geq 0}$ The large deviations or differences in these increments ensure that P_i *are* large enough to stimulate oscillation in equation (2.1). Since the oscillatory results of the solutions of the stochastic delay differential equation (1.3) will often be compared with those of the deterministic delay differential equation (1.4), it becomes necessary to reduce the deterministic delay differential equation (1.4) to a pure delay differential equation in terms of z. We do this by setting $x(t) = z(t)e^{at}$ to equation (1.4), that is,

$$x'(t) = z'(t)e^{at} + az(t)e^{at}$$

$$z'(t)e^{at} + az(t)e^{at} = az(t)e^{at} + \sum_{i=1}^{n} b_i z(t - r_i) e^{a(t-r_i)}$$

$$z'(t)e^{at} = \sum_{i=1}^{n} b_i z(t - r_i) e^{at}.e^{-ar_i}$$

$$z'(t)e^{at} = \sum_{i=1}^{n} b_i z(t - r_i) e^{-ar_i} \tag{2.3}$$

The proof of the main result relies upon invoking for use, on a path-wise basis, that is (for each w in some almost sure subset $\Omega^* \subseteq \Omega$) to Equation (2.1). The result below which concerns oscillatory properties of solutions is extracted from Li [15], (Theorem 2).

Proposition 2.1:

Let $\Gamma_n = \max\{r_1, r_2, .., r_n\}$. *Suppose that* $\sum_{i=1}^{n}\int_{h(t)}^{t} P_i(s)ds > 0 \;\; for \;\; t \geq t_0$, *for some* $t_0 > 0$ *and that* $\underset{t\to\infty}{Lim\,Sup}\int_{t-r_n}^{t_n} P_n(s)ds > 0$ *. If in addition,*

$$\int_{t_0}^{\infty}\left(\sum_{i=1}^{n} P_i(t)\right) In\left(e\sum_{i=1}^{n}\int_{h(t)}^{t} P_i(s)ds\right)dt = \infty. \qquad (2.4)$$

where h:[t_0, ∞) →$\Re^+$ is a non-decreasing continuous function satisfying $h(t) < t$, $h(t) \to \infty$ as $t \to \infty$. Then every solution of

$$Z'(t) = -\sum_{i=1}^{n} P_i(t) Z(h(t)) \qquad (2.5)$$

oscillates.

We also have results pertaining to non- oscillatory solutions of Equation (2.5). The following is found in Ladde et al [14] (Theorem 2.7.4)

Proposition 2.2:

Assume that

$$\int_{t-r_n}^{t_{i}} \sum_{i=1}^{n} P_i(s)ds < \frac{1}{e} \qquad (2.6)$$

Then equation (2.5) has an eventually positive solution and hence non-oscillatory.

***Remark* 2.1**

We will show that if $h(t) = t - r_i$ satisfies the condition of proposition (2'1), then the solution of the SDDE (1.3) is almost certainly(*a.c*) oscillatory. Also we comment that by using proposition (2.2), where the noise perturbation is absent, the deterministic delay differential equation (1.4) can have a non-oscillatory solution. The pair of oscillatory and non-oscillatory results chosen from the deterministic theory of oscillation as in proposition 1 and proposition 2 applies directly to the random delay differential equation (2.1)

We need the following Lemmas to prove the main results. The following is a special case extracted form Elabbasy et al [10] (Lemma 1.3).

***Lemma* 2.1**

If $\underset{t\to\infty}{Lim\,Sup}\int_{t}^{t+r_i} P_i(s)ds > 0$ *for some i and x(t) is an eventually positive solution of* $x'(t) = -\sum_{i=i}^{n} P_i(t)x(t - r_i)$, *then for the same i,*

$$\lim_{t\to\infty} Sup \frac{x(t - r_i)}{x(t)} < \infty \qquad (2.7)$$

The following is found in Li [15] (Lemma 2)

***Lemma* 2.2**

Consider the delay differential equation $x'(t) = -\sum_{i=1}^{n} P_i(t)x(t - r_i), \;\; t \geq t_0$

where $P_i \geq 0$ *are continuous and* $r_i > 0$ *are constants. If the equation has an eventually positive solutions, then*

$$\int_{t}^{t+r_i} P_i(s)ds \leq 1, \;\; i = 1, 2, ..., n \qquad (2.8)$$

eventually.

The following conjugation relationship is a special case of the results found in Lisei [16] (Theorem 3.5):

***Lemma* 2.3:**

Consider the stochastic functional differential equation driven by continuous helix spatial Semi-martingale of the Kunita type

$$\left.\begin{array}{l} dX(t) = H(X, X(t), X_t) + M(dt, X(t)),\ t \geq 0 \\ X(0) = v \in \Re^d,\ X_0 = \eta \in L^2([-r,0],\Re) \end{array}\right\} \quad (2.9)$$

where $X_t = X(t - r, w)$, $t \geq 0$, $w \in \Omega$ and a random functional differential equation of the form

$$\left.\begin{array}{l} dY(t) = G(t, Y(t), Y_t)dt \\ Y(0) = v \in \Re^d,\ Y_0 = \eta \in L^2([-r,0],\Re) \end{array}\right\} \quad (2.10)$$

Also let $\{\Lambda(t,w)\}_{t\geq 0}$ be a stationary bijective random process. Let $\{X(t,w)\}_{t\geq 0}$ be the solution of the stochastic functional differential equation and $\{Y(t,w)\}_{t\geq 0}$ be solution of the random functional differential equation. If for $t \geq -r$, we define

$$Y(t) = X(t)\Lambda^{-1}(t) \quad (2.11)$$

Then the following transformation or conjugation relationship holds:

$$X(t) = Y(t-r,w)\Lambda(t-r,w)\Lambda^{-1}(t,w), \quad for\ t \geq 0, w \in \Omega \quad (2.12)$$

2.2.1 The Transformation of Solution

In order to prove the existence of oscillatory and non-oscillatory solutions of the stochastic delay differential equation (1.3), the solution X of the SDDE is decomposed into a product of a nowhere differentiable but positive geometric process $\eta(t)$ with well understood properties and a process $Z(t)$ with a continuously differentiable sample paths which solves the scalar random delay differential equation (2.1)

To this end, we define a strictly positive process; $\{\eta(t)\}_{t\geq -r}$, which satisfies $\eta(t) = 1$, for $t \in [-r, 0]$ and also satisfies for all $t \geq 0$, $\eta(t) = \exp\left[\left(a - \frac{\mu^2}{2}\right)t + \mu B(t)\right]$ That is, $\eta(t)$ solves

$$\left.\begin{array}{l} d\eta(t) = a\eta(t)dt + \mu\eta(t)dB(t),\ t > 0 \\ \eta(t) = 1, \quad t \in [-r,0] \end{array}\right\} \quad (2.13)$$

We also call the almost sure subset on which η exists $\Omega^* \subseteq \Omega$ with $P[\Omega^*] = 1$. We again define for all $t \geq -r$ the process

$$Z(t) = X(t)/\eta(t) \quad (2.14)$$

where $X(t)$ is the solution of Equation (1.3). Moreover, the process $Z(t)$ is well defined since η is an entirely positive process and its properties are well known. This transformation in equation (2.14) above is important because it builds a relationship between Z and the solution X of the SDDE (1.3). The zeros of the process Z correspond to the zeros of the process X. Hence it is sufficient to analyze the oscillatory behaviour of Z in order to determine the oscillatory properties of X. This approach is of great benefit in the sense that there is a set of deterministic results (as we have in proposition 2.1 and proposition 2.2) that apply directly to the sample paths of the solution $Z(t)$ of the random DDE (2.1). This technique was applied in Appleby and Buckwar [3]. We comment here that many similar results, for example of equations with a single constant delay and equations with positive and negative coefficients, exist in the deterministic literature. These results could be used together with the technique in Appleby and Buckwar [3] to develop more general results pertaining to oscillatory and non-oscillatory behaviour of solutions of stochastic delay differential equations. Applying the result of Lemma 2.3 (or the stochastic integration by parts as in [3] to (2.7) and (2.8)) yields

$$Z(t) = Z(0) + \int_0^t \sum_{i=1}^n b_i Z(s - r_i)\eta(s - r_i)\eta(s)^{-1} ds,\ t \geq 0 \quad (2.15)$$

From the continuity of the integrand on the right hand side of (2.15), we see that $Z \in C((0, \infty), \Re)$, the space of all continuous functions that are once differentiable from the interval $(0, \infty)$ to $\Re$ and hence Equation (2.15) can be written as

$$Z'(t) = \sum_{i=1}^n b_i Z(t - r_i)\eta(t - r_i)\eta^{-1}(t) \quad (2.16)$$

Equation (2.16) is only a symbolic representation of Equation (2.15). Hence the solution $\{X(t)\}_{t\geq 0}$ of the stochastic delay differential equation (1.3) can be characterized using the following Lemma. It is a special case of the result extracted from Chilarescu et al [9](Theorem 2.2):

***Lemma* 2.4**

Consider

$$\left.\begin{aligned}\mathrm{dX}(\mathrm{t}) &= \left[aX(t)+\sum_{i=1}^{n} b_i X(t-r_i)\right]dt + \mu X(t)dB(t) \\ X(t) &= \psi(t), \qquad t\in[-r,0]\end{aligned}\right\} \tag{2.17}$$

If $\eta(t)$, $t \geq -r$ is a process which satisfies $\eta(t) = 1$ for $t \in [-r,0]$ and $\eta(t)=\exp\left[\left(a-\frac{\mu^2}{2}\right)t+\mu\mu dB(t)\right]$ for all $t \geq 0$. *Then the solution $X(t)$ of equation (2.17) is unique and is given by;*

$$X(t) = \eta(t)\left[X(0)+\int_0^t \sum_{i=1}^{n} b_i X(s-r_i)\eta(s)^{-1} ds\right]. \tag{2.18}$$

Moreover, the solution (2.18) has the following property

$$E\left[\underset{-r\leq s\leq t}{Sup}|P(s)|^2\right] \leq \left[1+2e^{\left(2a-\mu^2\right)t}\right]E\left[\underset{-r\leq u\leq 0}{Sup}|\psi(u)|^2\right]e^{\lambda t} \text{ where } \lambda = \frac{2b_i\left[e^{\left(2a-\mu^{2^2}\right)}-1\right]}{2a-\mu^2}.$$

3 0 The main results

In the result below, we establish that whenever the feedback intensity is negative and for whatever selection of initial function, the solution of the SDDE(1.3) is almost certainly oscillaory.

***Theorem* 3.1**:

Assume that $0 < r_i \leq r < \infty$ and $t \to t - r_i$.is non-decreasing If $b_i < 0$. Then for every continuous initial function ψ, equation (1.3) has an oscillatory solution on $[0,\infty)$ almost certainly.

Proof

By the relationship in equation (2.14), and since $\eta(t)$ is strictly positive, the set W = $\{t > 0: X(t) = 0\}$can only satisfy sup $W = +\infty$ if and only if the set $W^* = \{t \geq 0: Z(t) = 0\}$ satisfies Sup$W^* = 0$ following from the definition of oscillation of a non-trivial continuous function. Since $\{X(t)\}_{t\geq 0}$ is defined on the probability triple (Ω, F, P), we define for $t\geq 0$, $w \in \Omega$ $P_i(t,w) = -b\eta(t-r_i,w)\eta^{-1}(t,w)$. Then $P_i(.)$ is an almost certainly positive continuous function on $[0,\infty)$. Moreover, Z satisfies the equation

$$Z'(t,w) = -P_i(t,w)Z(t-r_i,w), \; t>0 \tag{3.1}$$

Hence it satisfies condition (2.4) of proposition 2.1 and thus almost certainly oscillatory, if not equation (2.1) may have an almost sure positive solution $Z(t)$ which is in fact non-increasing. We define $\alpha(t) = -Z'(t)/Z(t)$, then $\alpha(t)$ is non-negative and continuous and by definition, there exists $t_1 \geq t_0$ such that $Z(t_1) > 0$ for all $t \geq t_1$.such that $Z(t) = Z(t_1)\exp\left(-\int_{t-\rho_1}^{t}\alpha(s)ds\right)$, *for* $t \geq t_1$.Moreover, α(t) satisfies the generalized characteristic equation

$$\alpha(t) = \sum_{i=1}^{n} P_i(t)\exp\left(\int_{t_1-\rho}^{t}\alpha(s)ds\right) \tag{3.2}$$

The following properties of the exponential function are easily established. Assume that $x \geq 0$

$$\left.\begin{aligned}(i)\ e^{rx} &\geq re^{x}+1-r, \quad if\ \ r\geq 1\\ (ii)\ \ e^{rx} &\leq re^{x}+1-r, \quad if\ \ r<1\\ (iii)\ e^{rx} &\geq x+\frac{In(er),}{r} \quad if\ \ r>0\end{aligned}\right\} \tag{3.3}$$

Using (*iii*) in (3.3) above, we have

$$\alpha(t)=\sum_{i=1}^{n}P_i(t)\exp\left(C(t).\frac{1}{C(t)}\int_{t-r_{I_i}}^{t}\alpha(s)ds\right)_i \geq \sum_{i=1}^{n}P_i(t)\exp\left(\frac{1}{C(t)}\int_{t-r_{I_i}}^{t}\alpha(s)ds+\frac{In(eC(t))}{C(t)}\right) \tag{3.4}$$

where

$$\begin{aligned}C(t)=\sum_{i=1}^{n}\int_{t-\rho_i}^{t}P_i(s)ds &= \left(\sum_{i=1}^{n}\int_{t_i}^{t+r_i}P_i(s)ds\right)\alpha(t)-\sum_{i=1}^{n}P_i(t)\int_{t-r_{i_i}}^{t_i}\alpha(s)ds\\ &\geq\left(\sum_{i=1}^{n}P_i(t)\right)\ In\left(e\sum_{i=1}^{n}\int_{t_i}^{t+r_i}P_i(s)ds\right)\end{aligned} \tag{3.5}$$

Now choose K, $T\in[0,\infty)$ with $K>T$. Then

$$\begin{aligned}\int_T^K\left(\sum_{i=1}^{n}\int_{t_i}^{t+r_i}P_i(s)ds\right)\alpha(t)dt-\sum_{i=1}^{n}\int_T^K P_i(t)\int_{t-r_{i_{i_i}}}^{t_i}\alpha(s)dsdt\\ \geq\int_T^K\left(\sum_{i=1}^{n}P_i(t)\right)\ In\left(e\sum_{i=1}^{n}\int_t^{t+r_i}P_i(s)ds\right)dt\\ \alpha(t)\int_{t-r_{tt_i}}^{t}P_i(s)ds-P_i(t)\int_{t-r_{t_i}}^{t}\alpha(s)ds\geq P_i(t)In\left(e\int_t^{t+r_i}P_i(s)ds\right)\end{aligned} \tag{3.6}$$

By interchanging the order of integration, we obtain

$$\int_T^N\sum_{i=1}^{n}P_i(t)\int_{t-r_{i_i}}^{t}\alpha(s)dsdt\geq\sum_{i=1}^{n}\int_T^{K-r_i}\left(\int_s^{s+r_i}P_i(t)\alpha(s)dt\right)ds=\sum_{i=1}^{n}\int_T^{K-r_{i_i}}\alpha(t)\int_s^{s+r_i}P_i(t)dtds \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\sum_{i=1}^{n}\int_{K-r_{i_i}}^{K}\alpha(t)\int_t^{t+r_{i_i}}P_i(s)dsdt\geq\int_T^K\left(\sum_{i=1}^{n}P_i(t)\right)In\left(e\sum_{i=1}^{n}\int_t^{t+r_i}P_i(s)ds\right)dt \tag{3.8}$$

By Lemma 2.2, we have

$$\int_t^{t+r_i}P_i(s)ds\leq 1., \quad i=1,2,3,\ldots,n \tag{3.9}$$

eventually. Using (3.8) and (3.9)

$$\sum_{i=1}^{n}In\left(\frac{Z(K-r_i)}{Z(K)}\right)\geq\int_T^K\left(\sum_{i=1}^{n}P_i(t)\right)In\left(e\sum_{i=1}^{n}\int_t^{t+r_{i_i}}P_i(s)ds\right)dt \tag{3.10}$$

In view of condition (2.4) of proposition 1, we have

$$\underset{t\to\infty}{Lim}\prod_{i=1}^{n}\frac{Z(t-r_I)}{Z(t)}=\infty \tag{3.11}$$

But by lemma 1, we obtain $\lim_{t\to\infty}Sup\dfrac{x(t-r)}{x(t)}<\infty$, which contradicts (3.11). Hence $Z(.,w)$ is almost surely oscillatory and satisfies condition (3.13) of proposition 1. Suppose that there exists an almost sure set $\Omega^*\subset\Omega$ such that

$$\Omega^{*}=\left\{\begin{array}{l} w\in\Omega:\lim_{t\to\infty}Sup\int_{t-\Gamma_{nn}}^{t_n} P_n(s,w)ds>0,\ and \\ \int_{t_0}^{\infty}\left(\sum_{i=1}^{n}P_i(t)\right)In\left(e\sum_{i=1}^{n}\int_{h(t)}^{t}P_i(s,w)ds\right)dt=\infty \end{array}\right\} \text{ with } P[\Omega^{*}]=1$$

Then as $P_i(.)$ and $h(t) = t- r_i$ satisfy the hypothesis of proposition 2.1, It follows that for each $w\in\Omega^*$ the trajectory $Z(.,w)$ is oscillatory and so the path $X(.,w)$ is oscillatory and hence as Ω^* is an almost sure set, it follows that the solution $X(t)$ of the SDDE (1.3) is almost certainly oscillatory. By the properties of the random function $P_i(.)$, we observe that

$$\int_{t}^{t+r_{i_{i}}} P_i(s)ds \;=\int_{t}^{t+r_i} -b\exp\left(-(a-\frac{\mu^2}{2})\right)r_i\exp(-\mu(B(s)-B(s-r_i)))ds$$

$$\geq -b(\max\left(1,\exp\left(-\left(a-\frac{\mu^2}{2}\right)\right)r_i\right)\int_{t-r_{i_i}}^{t}\exp(-\mu(B(s)-B(s-r_i)))ds$$

It is observed (See [3]) that the sure event $\Omega^*\subseteq\Omega$ as defined above exists eventually whenever

$$\limsup_{t\to\infty}\int_{t-r_{i_i}}^{t}\exp(-\mu(B(s)-B(s-r_i)))ds=\infty. \tag{3.12}$$

and hence (1.3) has an oscillatory solution almost certainly.

4.0 Conclusion

In the stochastic delay differential equation (1.3), under theorem 2, the important factor that stimulates oscillation is equation (3.12), which must always occur in the stochastic case as a result of the presence of the multiplicative noise. If r_i are small enough, the integral in (3.12) is made so small that the condition in proposition 2 holds in the deterministic case (1.4) and at that instant, a non-oscillatory solution occurs in (1.4) but this cannot happen in the SDDE (1.3) as a result of (3.12). Hence the multiplicative noise sustains oscillation in the stochastic case (1.3) even when the non-stochastic equation (1.4) has a non-oscillatory solution. Therefore, the multiplicative noise stimulates oscillation about the zero equilibrium solution which may not necessarily be present in the deterministic case where $\mu = 0$. It should be noted that the noise has not entirely replaced the time delay as the cause of the oscillation. However, the time delay is no longer the sole factor in the oscillatory phenomenon of the stochastic delay differential equation.

The following result shows that the crucial condition (3.12) which ensures oscillation in the random equation must always hold in the stochastic case so that all solutions if the SDDE (1.3) are almost certainly oscillatory. It is a special case in Appleby [3]

***Lemma* 4.1**

Assume that $r_i\in\Re^+$ and satisfies $0<r_i\leq r<\infty$. If $\mu\neq 0$, then $\limsup_{t\to\infty}\int_{t-r_{i_i}}^{t}\exp(-\mu(B(s)-B(s-r_i)))ds=\infty$ almost certainly hold.

In the result below, we prove that for positive feedback and any continuous initial function $\psi\in([-r,0],\Re^+)$, every solution of equation (1.3) is eventually positive.

***Corollary* 4.2**

Assume that $b_i>0$ and $\psi(t)\in([-r,0],\Re^+)$.for $t\in[-r,0]$. If $0<r_i\leq r<\infty$, then equation (1.3) *has an eventually positive solution on* $[0,\infty)$ *and hence non-oscillatory.*

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