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Improved continuous method for direct solution of general second order ordinary differential equation

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Abstract

In this paper a numerical method of solving general second order initial value problems with step length k = 4 is developed. The approach is based on the collocation of the differential system and interpolation of the approximate solution. Some predictors for calculating the unknown in the corrector were also derived. The efficiency of this scheme was tested with some numerical examples.

***Keywords*:** Collocation, interpolation, approximate solution, predictors, correctors, step length.

1.0 Introduction

Second order ordinary differential equations of the form

$$y'' = f(x, y(x), y'(x)), \quad y(0) = \eta_0, \quad y'(a) = \eta_1 \tag{1.1}$$

have a wide area of application among which is satellite tracking, celestial mechanics, mass action kinetics, solar equation, molecular biology, and spatial discretization of hyperbolic differential equations [1].

Attempts to solve equation (1.1) have remained a problem. Commonest among the methods used in solving (1.1) is the reduction of the system to first order ordinary differential equation and developing appropriate numerical methods to solve the resulting system [3]. The major constraint of this method is the need to develop separate computer subprogram to initialize the starting value for evaluating the function arising from the system. The validity of such a method depends, to a large extent, on the need to cope with the time and cost of getting reliable results. This is a serious hindrance, though it ironically doubles as an incentive to investigating into cheaper and faster approaches that the attention has been given to other researchers.
[5] proposed the concept of derivation of Linear Multistep Methods (LMM) in terms of power series involving finite difference operator of solving (1.1). [2] worked on linear multistep method by collocation in which Numerov method was reaffirmed.

There are many other contributors whose methods have symbols and functions evaluation per iteration resulting in complexity of any serious practical application. The computational effort required for a numerical method is measured by the total number of function evaluation over the total number of integration steps. Although, the efficiency of such methods is reduced, if the number of output points become very large.

Regrettably, little has been done in considering linear multistep method with continuous coefficient as a means of solving equation (1.1). Linear multistep method with continuous coefficient directly solves both initial and boundary value problems of general differential equations. In this paper, we propose a collocation procedure which leads to a continuous scheme and their derivatives. The scheme developed in this paper is a linear multistep one, which is capable of solving consistent variable coefficient problems [4].

2.0 Methodology

We define a basis function in the form of power series, where

$$y(x)=\sum_{j=0}^{2k-1} a_j\psi_j(x) \qquad (2.1)$$

where $\psi_j=\psi^j$ and a_j's are constants to be determined. The first and second derivatives of (2.1) give

$$y'(x)=\sum_{j=0}^{2k-1} ja_j\psi_{j-1}(x) \qquad (2.2)$$

$$y''(x)=\sum_{j=0}^{2k-1} j(j-1)a_j\psi_{j-2}(x) \qquad (2.3)$$

Collocating (2.3) at grid point $x=x_{n+j},\ 0\le j\le k$ where $k=4$ and interpolate (2.1) at $x=x_{n+j},\ 2\le j\le 4$ give the following:

$$f_{n+1}=\sum_{j=0}^{2k-1} j(j-1)a_j\psi_{j-2}(x_{n+j}),\ 0\le j\le 4 \qquad (2.4)$$

$$y_{n+j}=\sum_{j=2}^{2k-1} a_j\psi_j(x_{n+j}),\ 2\le j\le 4 \qquad (2.5)$$

Using Gaussian elimination method to evaluate the a_j's and evaluating at x_{n+4}, an equation of the form

$$y(x)=\sum_{j=0}^{2k-1}\alpha_j(x)y_{n+j}+h^2\sum_{j=0}^{2k-1}\beta_j(x)f_{n+j} \qquad (2.6)$$

is obtained where $f_{n+j}=f(x_{n+j},y_{n+j},y'_{n+j})$

If $t=\dfrac{x-x_{n+3}}{h}$ is substituted into (2.6), the results is the continuous function given as

$$\alpha_2(t)=t$$

$$\alpha_3(t)=-(t-1)$$

$$\beta_0(t)=\frac{h^2}{1440}(2t^6+6t^5-5t^4-20t^3+11t)$$

$$\beta_1(t)=\frac{h^2}{360}(-2t^6-9t^5+5t^4+30t^3+18t)$$

$$\beta_2(t)=\frac{h^2}{240}(24t^6+12t^5-5t^4-60t^3+55t)$$

$$\beta_3(t)=\frac{h^2}{360}(-2t^6-15t^5-25t^4+50t^3+18t)$$

$$\beta_4(t)=\frac{h^2}{1440}(2t^6+18t^5+55t^4+60t^3 21t) \qquad (2.7)$$

The first derivatives give

$$\alpha_2(t)=\frac{1}{h}$$

$$\alpha_3(t)=-\frac{1}{h}$$

$$\beta_0(t) = \frac{h}{1440}(12t^5 + 30t^4 - 20t3 - 60t^2 + 11)$$

$$\beta_1(t) = \frac{h^2}{360}(-12t^5 - 25t^4 + 20t^4 + 90t^2 + 55)$$

$$\beta_2(t) = \frac{h}{240}(144t^5 - 60t^4 + 20t^3 - 180t^2 + 55)$$

$$\beta_3(t) = \frac{h}{360}(-2t^5 - 75t^4 - 100t^3 + 150t^2 + 360t + 118)$$

$$\beta_4(t) = \frac{h}{1440}(12t^5 + 90t^4 + 220t^3 + 180t^2 - 21) \qquad (2.8)$$

Evaluating at $t = 1$ gives the discrete scheme

$$y_{n+4} - 2y_{n+3} + y_{n+2} = \frac{h^2}{240}(19 f_{n+4} + 204 f_{n+3} + 14 f_{n+2} + 4 f_{n+1} - f_n) \qquad (2.9)$$

The scheme is of order 6 and error constant 0.00418. Interval of absolute stability, $X(\theta) = (-5.758, 0)$. Predictors for calculating y_{n+2} and y_{n+1} are derived using Taylor's series expansion.

The following predictors and their derivatives have been developed to calculate $y_{n+2}, y'_{n+2}, y_{n+3}, y'_{n+3}, y_{n+4}$ and y'_{n+4} in (2.9) respectively. Thus

$$y_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+1} \qquad (2.10)$$

$$y'_{n+2} = \frac{y_{n+1}}{h} + \left(\frac{3}{2}\right) h f'_{n+1} \qquad (2.11)$$

Similarly,

$$y_{n+3} = 2y_{n+2} - y_{n+1} + \frac{h^2}{12}(13 f_{n+2} - 2 f_{n+1} + f_n) \qquad (2.12)$$

$$y'_{n+3} = (y_{n+2} - y_{n+1}) + \frac{h}{24}(53 f'_{n+2} - 26 f'_{n+1} + 9 f'_n) \qquad (2.13)$$

Finally,

$$y_{n+4} = 2y_{n+3} - y_{n+2} + \frac{h^2}{12}(14 f_{n+1} - 5 f_{n+2} + 4 f_{n+1} - f_n) \qquad (2.14)$$

and

$$y'_{n+4} = (y_{n+3} - y_{n+2}) + \frac{h}{360}(922 f'_{n+3} - 771 f'_{n+2} + 516 f'_{n+1} - 7 f'_n) \qquad (2.15)$$

The predictor for y_{n+1} and y'_{n+1} can be seen in [2].

2.3 Numerical examples

We solve the following example to illustrate our method (2.9).

Problem I: $y'' = x(y')^2 = 0, \quad y(0) = 1,\ y'(0) = \frac{1}{2}.$

Exact solution: $y(x) = 1 + \frac{1}{2} In\left(\frac{2+x}{2-x}\right)$

Problem II: $y'' = 2x^2 + \frac{1}{x^6} - \frac{2y'}{x} - \frac{y}{x^4}, \qquad y\left(\frac{2}{\pi}\right) = \frac{\pi^2}{4} - 1, \quad y'\left(\frac{2}{\pi}\right) = \frac{\pi^2}{4}(2-\pi)$

Exact solution: $y(x) = 2\cos\left(\frac{1}{x}\right) - 3x\left(\frac{1}{x}\right) + \frac{1}{x^2}$.

Table 2.1: For problem I

x	$h = 1/30$	$h = 1/32$	$h = 1/40$	$h = 1/50$	$h = 1/60$
0.2	3.32D $^{-09}$	2.89D $^{-09}$	1.72D $^{-09}$	1.08D $^{-09}$	7.43D $^{-10}$
0.4	2.53D $^{-08}$	2.27D $^{-08}$	1.40D $^{-08}$	8.86D $^{-09}$	6.19D $^{-09}$
0.6	9.28D $^{-08}$	8.13D $^{-08}$	5.23D $^{-08}$	3.28D $^{-08}$	2.24D $^{-08}$
0.8	2.59D $^{-07}$	2.23D $^{-07}$	1.48D $^{-07}$	9.02D $^{-08}$	6.24D $^{-08}$
1.0	6.27D $^{-07}$	5.42D $^{-07}$	3.48D $^{-07}$	2.29D $^{-07}$	1.52D $^{-07}$

Table 2.2: For problem II

x	$h = 1/30$	$h = 1/32$	$h = 1/40$	$h = 1/50$	$h = 1/60$
1.2	1.19D $^{-04}$	1.05D $^{-04}$	6.068D $^{-05}$	4.23D $^{-05}$	2.92D $^{-05}$
1.4	1.44D $^{-04}$	1.26D $^{-04}$	8-05D $^{-05}$	5.11D $^{-05}$	3.53D $^{-05}$
1.6	1.61D $^{-04}$	1.42D $^{-04}$	9.04D $^{-05}$	5.75D $^{-05}$	3.97D $^{-05}$
1.8	1.74D $^{-04}$	1.53D $^{-04}$	9.77D $^{-05}$	6.22D $^{-05}$	4.30D $^{-05}$
2.0	1.84D $^{-04}$	1.62D $^{-05}$	1.03D $^{-05}$	6.58D $^{-05}$	4.53D $^{-05}$

3.0 Conclusion

The result clearly shows that the proposed scheme is suitable for solving general initial value problems of second order ordinary differential equation.

It is note worthy also, that methods with continuous coefficients have a lot of advantages over discrete schemes [4]. With continuous schemes, one can generate as many values as desired especially between the last two grid points. Derivatives of the continuous scheme to any possible order could be computed. This enables an n^{th} order ordinary differential equation to be solved directly without reduction to a system of first order equation.

A direct implication of this is that complicated computer programs are avoided. More so, our method can be used to construct global error estimate of (2.9). Most importantly, this method could be used to develop automatic codes for problems presented in (1.1).

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