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On coefficients in the partial fractions of some trigonometric and exponential functions

[1]O. Ogbereyivwe, [2]K. Emumejaye, and [3]K. I. Ekeh
[1,2,3] Department Of Mathematics and Basic Sciences,
Delta State Polytechnic, Ozoro , Delta State, Nigeria.

Abstract

***Several works have been done in this area including Daniel S. and Tella Y.* [2]. *In this paper, we consider resolving rational functions containing trigonometric and exponential functions in their denominators into the sum of its partial fractions equivalent where all the unknown constants and coefficients are obtained by recursive method; an extension of* [2]. *Also, a general formula for obtaining these coefficients as n tends to infinity is obtained*.**

1.0 Introduction

Suppose that $\frac{f(x)}{g(x)}$ defines a rational function, so that $f(x)$ and $g(x)$ are polynomials, and suppose that the degree of $f(x)$ is less than the degree of $g(x)$. If $g(x)$ can be factorised into a product of some different linear factor, each to some index, and some different irreducible quadratic factors, each to some index, then $\frac{f(x)}{g(x)}$ can be written as a sum of terms. [1], [4].

Resolving rational functions with trigonometric and exponential functions into sum of simpler rational functions is stimulating particularly in the area of calculus and applied Mathematics that may result to problems of this nature we are considering. In this research, we consider cases where the numerator is a constant unit function and the denominators as functions of $\sin x$ and e^x, the case of $\cos x$ was considered in [2]. In each case, the coefficients are obtained by recursive method derived explicitly.

2.0 The case $R(x) = \dfrac{1}{(1-x^2)\sin x}$

Consider,

$$R(x) = \frac{1}{(1-x^2)\sin x} = \frac{1}{(1-x^2)\sum_{i=0}^{n}\frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

$$= \frac{1}{(1-x^2)\sum_{i=1}^{n}\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}} = \frac{1}{(1-x^2)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \Lambda + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}\right)} \quad (2.1)$$

$$= \frac{1}{x(1-x^2)\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \Lambda + \frac{(-1)^{n-1}x^{2n-2}}{(2n-1)!}\right)} \quad (2.2)$$

We resolve (2.2) into partial fraction for the following values of *n*.
When *n* = 1.

$$R(x)=\frac{1}{x(1-x^2)}\equiv\frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x}$$

[1]Corresponding author: e-mail: vesenelly@yahoo.com Phone Number: 0805-278-6250

$x=1\ \Rightarrow\ C=\frac{1}{2},$

$x=-1\ \Rightarrow\ B=-\frac{1}{2}$

$x=0\Rightarrow A=1$

$\therefore\quad R(x)=\frac{1}{2(1-x)}+\frac{1}{2(1+x)}+\frac{1}{x}$

When $n=2$

$$R(x)=\frac{1}{(1-x^2)\left(x-\frac{x^3}{3!}\right)}=\frac{1}{x(1-x^2)\left(1-\frac{x^2}{3!}\right)}\equiv\frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x}+\frac{D_0+D_1x}{1-\frac{x^2}{3!}}$$

$$\equiv A(1-x^2)\left(1-\frac{x^2}{3!}\right)+Bx(1-x)\left(1-\frac{x^2}{3!}\right)+Cx\left(1-\frac{x^2}{3!}\right)+x(D_0+D_1x)(1-x^2)$$

$$\equiv A\left(1-\frac{x^2}{3!}-x^2+\frac{x^4}{3!}\right)+Bx\left(1-\frac{x^2}{3!}-x+\frac{x^3}{3!}\right)+Cx\left(1-\frac{x^2}{3!}+x-\frac{x^3}{3!}\right)+x(D_0+D_0x^2+D_1x-D_1x^3)$$

$$x=0\Rightarrow A=1. \tag{2.3}$$

$x=1\quad\Rightarrow\quad 2C\left(1-\frac{1}{3!}\right)=1$

Therefore,

$$C=-\frac{1}{2\left(1-\frac{1}{3!}\right)}=k \tag{2.4}$$

$x=-1\Rightarrow$

$$B=-\frac{1}{2(1-\frac{1}{3!})}=-k \tag{2.5}$$

For the coefficient of x, we have

$0=B+C+D_0$, but $B=-C\Rightarrow$

$$D_0=0 \tag{2.6}$$

For the coefficient of x^2 we have:

$$-\frac{1}{3!}A-A-B+C+D_1=0$$

$2k-\frac{1}{3!}-1+D_1=0\Rightarrow$

$$D_1=1+\frac{1}{3!}-2k \tag{2.7}$$

When $n=3$,

$$R(x)=\frac{1}{x(1-x^2)\left(1-\frac{x^2}{3!}+\frac{x^3}{5!}\right)}$$

$$\equiv\frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x}+\frac{D_0+D_1x+D_2x^2}{\left(1-\frac{x^2}{3!}+\frac{x^3}{5!}\right)}$$

$$1\equiv A(1-x^2)\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}\right)+Bx(1-x)\left(1-\frac{x^2}{3!}+\frac{x^3}{5!}\right)+Cx(1+x)\left(1-\frac{x^2}{3!}+\frac{x^3}{5!}\right)$$
$$+(D_0+D_1x+D_2x^2)(1-x^2)x$$

$$1\equiv A\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-x^2+\frac{x^4}{3!}\right)+Bx\left(1-\frac{x^2}{3!}+\frac{x^3}{5!}-x+\frac{x^3}{3!}-\frac{x^4}{5!}\right)+Cx\left(1-\frac{x^2}{3!}+\frac{x^3}{5!}+x-\frac{x^3}{3!}+\frac{x^4}{5!}\right)$$
$$+x(D_0+D_1x+D_2x^2-D_0x^2-D_1x^3-D_2x^4)$$

$x=0\Rightarrow$

$$A=1 \tag{2.8}$$

$x\ =\ 1\quad\Rightarrow\quad 2C\left(1-\frac{1}{3!}+\frac{1}{5!}\right)\ =1$

$\Rightarrow$ $$C = \frac{1}{2(1-\frac{1}{3!}+\frac{1}{5!})} = k \tag{2.9}$$

x = -1 $\Rightarrow$ $$B = -\frac{1}{2(1-\frac{1}{3!}+\frac{1}{5!})} = -k \tag{2.10}$$

For the coefficient of *x*, $B + C + D_0 = 0$, *But* $B = -C$

$\Rightarrow$ $$D_0 = 0 \tag{2.11}$$

For the coefficient of x^2 $$-\frac{1}{3!}A - A - B + C + D_1 = 0$$

$$2k - \frac{1}{3!} - 1 + D_1 = 0$$

$$D_1 = -2k + 1 + \tfrac{1}{3!} \tag{2.12}$$

Coefficient of x^3

$$-\frac{1}{3!}B - \frac{1}{3!}C + D_2 - D_0 = 0$$

$\Rightarrow$ $$D_2 = D_0 = 0 \tag{2.13}$$

Coefficient of x^4

$$\frac{1}{5!}A + \frac{1}{3!}A + \frac{1}{5!}B - \frac{1}{3!}C + D_3 - D_1 = 0$$

$$\frac{1}{5!} + \frac{1}{3!} - \frac{2}{3!}k + D_3 - D_1 = 0$$

$$D_3 = \frac{2}{3!}k - \frac{1}{3!} - \frac{1}{5!} + D_1 \tag{2.14}$$

When $n = 4$ $$R(x) = \frac{1}{x(1-x^2)\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right)}$$

$$1 \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D_0 + D_1x + D_2x^2 + D_3x^3 + D_4x^4 + D_5x^5}{\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right)}$$

$$1 \equiv A(1-x^2)\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right) + Bx(1-x)\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right)$$
$$+ Cx(1+x)\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right) + x(1-x^2)\left(D_0 + D_1x + D_2x^2 + D_3x^3 + D_4x^4 + D_5x^5\right)$$

$$x = 0 \Rightarrow A = 1$$

$x = 1$ $\Rightarrow$ $$2C\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}\right) = 1$$

$$C = \frac{1}{2\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}\right)} = k \tag{2.15}$$

$x = -1$ $\Rightarrow$ $$B = -\frac{1}{2C\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!}\right)} = -k \tag{2.16}$$

Coefficient of *x*, $B + C + D_0 = 0 \Rightarrow D_0 = 0$ (2.17)

Coefficient of x^2 $$-\frac{1}{3!}A - A - B + C + D_1 = 0$$

$$-\frac{1}{3!}+2k-B+D_1=0$$

$$D_1=-2k+\frac{1}{3!}+1 \qquad (2.18)$$

Coefficient of x^3

$$-\frac{1}{3!}B-\frac{1}{3!}C+D_2-D_0=0$$

$$D=D_0=0 \qquad (2.19)$$

Coefficient of x^4

$$\frac{1}{5!}A+\frac{1}{3!}A+\frac{1}{3!}B-\frac{1}{3!}C+D_3-D_1=0$$

$$\frac{1}{5!}+\frac{1}{3!}-\frac{2}{3!}k+D_3-D_1=0$$

$$D_3=\frac{2k}{3!}-\frac{1}{5!}-\frac{1}{3!}+D_1 \qquad (2.20)$$

Coefficient of x^5

$$\frac{1}{5!}B+\frac{1}{5!}C-D_2=0$$

$\Rightarrow$

$$D_2=0 \qquad (2.21)$$

Coefficient of x^6

$$-\frac{1}{7!}A-\frac{1}{5!}A-\frac{1}{5!}B+\frac{1}{3!}C+D_5-D_3=0$$

$$-\frac{1}{7!}-\frac{1}{5!}+\frac{2}{5!}k+D_5-D_3=0$$

$$D_5=-\frac{2}{5!}k+\frac{1}{7!}+\frac{1}{5!}+D_3 \qquad (2.22)$$

When $n=5$, $R(x)=\dfrac{1}{x(1-x^2)\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\right)}$

$$1\equiv\frac{A}{x}+\frac{B}{1+x}+\frac{C}{1-x}+\frac{D_0+D_1x+D_2x^2+D_3x^3+D_4x^4+D_5x^5+D_6x^6+D_7x^7}{\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\right)}$$

$$1\equiv A(1-x^2)\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\right)+Bx(1-x)\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\right)$$

$$+Cx(1+x)\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}\right)$$

$$+x(1-x^2)(D_0+D_1x+D_2x^2+D_3x^3+D_4x^4+D_5x^5+D_6x^6+D_7x^7)$$

$$A=1 \qquad (2.23)$$

$x=1\Rightarrow$

$$2C\left(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\frac{1}{9!}\right)=11$$

$$C=\frac{1}{2\left(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\frac{1}{9!}\right)}=k \qquad (2.24)$$

$x=-1\Rightarrow$

$$B=\frac{1}{2\left(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\frac{1}{9!}\right)}=k \qquad (2.25)$$

Coefficient of x $\quad B + C + D_0 = 0 \quad \Rightarrow \quad D_0 = 0$ (2.26)

Coefficient of x^2 $\quad \frac{1}{3!}A - A - B + C + D_1 = 0$

$$-\frac{1}{3!} - 1 + 2k + D_1 = 0$$

$$D_1 = -2k + \frac{1}{3!} + 1 \qquad (2.27)$$

Coefficient of x^3 $\quad \frac{1}{3!}B - \frac{1}{3!}C + D_2 - D_0 = 0$

$\Rightarrow$ $\quad D_2 = D_0 = 0$ (2.28)

Coefficient of x^4 $\quad \frac{1}{5!}A + \frac{1}{3!}A + \frac{1}{3!}B - \frac{1}{3!}C + D_3 - D_1 = 0$

$$\frac{1}{5!} + \frac{1}{3!} - \frac{2}{3!}k + D_3 - D_1 = 0$$

$$D_3 = \frac{2}{3!}k - \frac{1}{5!} - \frac{1}{3!} + D_1 \qquad (2.29)$$

Coefficient of x^5 $\quad \frac{1}{5!}B + \frac{1}{5!}C + D_4 - D_2 = 0$

$\Rightarrow$ $\quad D_2 = D_4 = 0$ (2.30)

Coefficient of x^6 $\quad -\frac{1}{7!}A - \frac{1}{5!}A - \frac{1}{5!}B + \frac{1}{3!}C + D_5 - D_3 = 0$

$$-\frac{1}{7!} - \frac{1}{5!} + \frac{2}{5!}k + D_5 - D_3 = 0$$

$$D_5 = -\frac{2}{5!}k + \frac{1}{7!} + \frac{1}{5!} + D_3 \qquad (2.31)$$

Coefficient of x^7 $\quad -\frac{1}{7!}B - \frac{1}{7!}C + D_6 - D_4 = 0$

$\Rightarrow$ $\quad D_6 = D_4 = 0$ (2.32)

Coefficient of x^8 $\quad \frac{1}{9!}A + \frac{1}{7!}A - \frac{1}{7!}C + D_7 - D_5 = 0$

$$\frac{1}{9!} + \frac{1}{7!} - \frac{2}{7!}k + D_7 - D_5 = 0$$

$$D_7 = \frac{2}{7!}k - \frac{1}{9!} - \frac{1}{7!} + D_5 \qquad (2.33)$$

Continuing in this manner we conclude as follows:

$$R_n(x) = \frac{1}{(1-x^2)\sin x} = \frac{1}{\left(1-x^2\right)\sum_0^n \frac{(-1)^n x^{2n+1}}{(2n+1)!}} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{\sum_{j=0}^{2n-2} D_j x^j}{\sum_0^n \frac{(-1)^n x^{2n}}{(2n+1)!}}$$

where

$$\begin{cases} D_1 = -2k+1+\frac{1}{3!},\ A=1, C=-B=k \\ D_{2j-2} = 0, \forall j = 1,2,3,\Lambda \\ D_{2j-1} = (-1)^{j-1}\frac{2k}{(2j+1)!} + \sum_{j}^{j+1}(-1)^{j+1}\frac{1}{(2j+1)!} + D_{2j-2}, \\ \forall j = 1,2,3,\Lambda \text{ , and } k = \frac{1}{2(1-\frac{1}{3!}+\frac{1}{5!}-\Lambda\ )} \end{cases} \tag{2.34}$$

3.0 The case of $\lim_{n\to\infty} R_n(x)$

Here, we observe the behavior of the coefficients as n tends to infinity. Thus we have as follows:

$$R_\infty(x) = \frac{1}{(1-x^2)\sin x} \equiv \frac{A}{x} + \frac{B}{1+x} + \frac{C}{1-x} + \lim_{n\leftarrow\infty}\left(\frac{\sum_{j=0}^{2n-2} D_j x^j}{\sum_0^n \frac{(-1)^n x^{2n}}{(2n+1)!}}\right)$$

But, $k_\infty = \frac{1}{\left(1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\Lambda\right)}$ =½sin1 = ½cosec1 and from (2.2) we have $C = -B =$ ½cosec1

where

$$\begin{cases} D_1 = -2k+1+\frac{1}{3!}, \quad A=1, \quad C=-B=\frac{1}{2}\cos ec\ 1 \\ D_{2j-2} = 0, \forall j = 1,2,3,\Lambda \\ D_{2j-1} = (-1)^{j-1}\frac{2\sin\ 1}{(2j+1)!} + \sum_{j}^{j+1}(-1)^{j+1}\frac{1}{(2j+1)!} + D_{2j-2}, \\ \forall j = 1,2,3,\Lambda \end{cases} \tag{3.1}$$

4.0 The case of $R_n(x) = \frac{1}{(1-x^2)e^x}$

The process is similar to that of section (2.0) with little modification. Thus;

$$R_n(x) = \frac{1}{(1-x^2)e^x} = \frac{1}{(1-x^2)(1+x+\frac{x^2}{2!}+\Lambda\ \Lambda\ +\frac{x^n}{n!}+\Lambda\ )} \quad \text{(see [1])}$$

$$= \frac{1}{(1-x^2)\sum_{j=0}^{n}\frac{x^j}{j!}} \tag{4.1}$$

We solve (4.1) for n = 1,2,3,…

When n = 1, we have $R_1(x) = \frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$. Thus,

$$A = B = \frac{1}{2} = k \tag{4.2}$$

When n = 2, we have $R_2(x) = \frac{1}{(1-x^2)(1+x)} = \frac{1}{(1-x\ \ )(1+x)^2} \equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{D_0}{(1+x)^2}$

$\Rightarrow$ $$A = \frac{-3}{4}, B = \frac{1}{4}, D_0 = \frac{1}{2} \quad (4.3)$$

When n = 3, we have $R_3(x) = \dfrac{1}{(1-x^2)\left(1+x+\dfrac{x^2}{2!}\right)} \equiv \dfrac{A}{1+x} + \dfrac{B}{1-x} + \dfrac{D_0 + D_1 x}{\left(1+x+\dfrac{x^2}{2!}\right)}$, hence we have

$$A = \frac{1}{2\left(\frac{1}{2!}\right)} = \alpha_3 \quad (4.4)$$

$$B = \frac{1}{2\left(2+\frac{1}{2!}\right)} = k_3 \quad (4.5)$$

$$D_0 = 1 - \alpha_3 - k_3 \quad (4.6)$$

$$D_1 = -2k_3 \quad (4.7)$$

When n = 4, we have $R_3(x) = \dfrac{1}{(1-x^2)\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}\right)} \equiv \dfrac{A}{1+x} + \dfrac{B}{1-x} + \dfrac{D_0 + D_1 x + D_1 x^2}{\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}\right)}$

Thus, we have the following $$A = \frac{1}{2\left(\frac{1}{2!}-\frac{1}{3!}\right)} = \alpha_4 \quad (4.8)$$

$$B = \frac{1}{2\left(2+\frac{1}{2!}+\frac{1}{3!}\right)} = k_4 \quad (4.9)$$

$$D_0 = 1 - \alpha_4 - k_4 \quad (4.10)$$

$$D_1 = -2k_4 \quad (4.11)$$

$$D_2 = D_0 - \left[\alpha_4\left(\frac{1}{2!}-1\right) + k_4\left(\frac{1}{2!}+1\right)\right] \quad (4.12)$$

When n = 5, we have

$$R_5(x) = \frac{1}{(1-x^2)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{3!}\right)} \equiv \frac{A}{1+x} + \frac{B}{1-x} + \frac{D_0 + D_1 x + D_2 x^2 + D_2 x^3}{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{3!}\right)}$$

Hence, we have the following:

$$A = \frac{1}{2\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)} = \alpha_5 \quad (4.13)$$

$$B = \frac{1}{2\left(2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\right)} = k_5 \quad (4.14)$$

$$D_0 = 1 - \alpha_5 - k_5 \quad (4.15)$$

$$D_1 = -2k_5 \quad (4.16)$$

$$D_3 = D_1 - \left[\alpha_5\left(\frac{1}{3!}-\frac{1}{2!}\right)+k_5\left(\frac{1}{3!}+\frac{1}{2!}\right)\right] \quad (4.17)$$

When $n = 6$, we have

$$R_5(x) = \frac{1}{(1-x^2)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{3!}+\frac{x^5}{3!}\right)} \equiv \frac{A}{1+x}+\frac{B}{1-x}+\frac{D_0+D_1x+D_2x^2+D_2x^3+D_2x^4}{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{3!}+\frac{x^5}{3!}\right)}$$

In like manner, we generate the following: $$A = \frac{1}{2\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)} = \alpha_6 \quad (4.18)$$

$$B = \frac{1}{2\left(2+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)} = k_6 \quad (4.19)$$

$$D_0 = 1-\alpha_6 - k_6 \quad (4.20)$$

$$D_1 = -2k_6 \quad (4.21)$$

$$D_4 = D_2 - \left[\alpha_6\left(\frac{1}{4!}-\frac{1}{3!}\right)+k_6\left(\frac{1}{4!}+\frac{1}{3!}\right)\right] \quad (4.22)$$

We conclude as follows: $$R_n(x) = \frac{1}{(1-x^2)e^x} = \frac{1}{(1-x^2)\sum_{j=0}^{n}\frac{x^j}{j!}} \equiv \frac{A}{(1+x)}+\frac{B}{(1-x)}+\frac{\sum_{j=0}^{n-1}D_jx^j}{\sum_0^n\frac{x^j}{j!}}$$

Where $n \le 2$, we have:

$$\begin{cases} A = B = \frac{1}{2}, \text{for } n = 1 \\ A = \frac{-3}{4}, B = \frac{1}{4}, D_0 = \frac{1}{2} \text{for } n = 2 \end{cases} \quad (4.23)$$

Where $n \ge 3$, we have;

$$\begin{cases} A = \alpha_n, B = k_n, n = 3,4,5,... \\ D_0 = 1-\alpha_n - k_n,\ D_1 = -2k_n,\ n = 3,4,5,\Lambda\ ,\alpha_n = k_n = 0, \forall\ n \le 2. \\ D_n = D_{n-2} - \left[\alpha_n\sum_{n-1}^{n}(-1)^{n-1}\frac{1}{n!}+k_n\sum_{n-1}^{n}\frac{1}{n!}\right] \\ \text{and } \alpha_n = \frac{1}{2\sum_{i=0}^{n}(-1)^{n-1}\frac{1}{n!}},\ k_n = \frac{1}{2\sum_{i=0}^{n}\frac{1}{n!}} \end{cases} \quad (4.24)$$

5.0 The case of $\lim_{n\to\infty}\left[R_n(x) = \frac{1}{(1-x^2)e^x}\right]$

$$R_\infty(x) =\equiv \frac{A}{(1+x)}+\frac{B}{(1-x)}+\frac{\sum_{j=0}^{+\infty}D_jx^j}{\sum_{n=0}^{+\infty}\frac{x^n}{n!}} \quad (5.1)$$

But, $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x$ (see [1]).

$$R_{\infty}(x) \equiv \frac{A}{(1+x)} + \frac{B}{(1-x)} + \frac{\sum_{j=0}^{+\infty} D_j x^j}{\lambda^x} \tag{5.2}$$

Also,

$$\begin{cases} \alpha_{\infty} = \dfrac{1}{2\sum_{n=0}^{\infty} (-1)^{n-1} \dfrac{1}{n!}} = \dfrac{1}{2e^{-1}} \quad , \\ k_{\infty} \quad = \dfrac{1}{2\sum_{n=0}^{\infty} \dfrac{1}{n!}} = \dfrac{1}{2e} \end{cases} \tag{5.3}$$

We conclude as follows as $n \to \infty$, we have:

$$\begin{cases} A = \alpha_{\infty}, B = k_{\infty}, n = 3,4,5,\Lambda \ ,\infty \\ D_0 = 1 - \alpha_{\infty} - k_{\infty}, D_1 = -2k_{\infty}, \quad n = 3,4,5,\Lambda \ ,\infty \\ D_{\infty} = D_{\infty-2} - \left[\alpha_{\infty} \sum_{n}^{\infty} (-1)^{n-1} \dfrac{1}{n!} + k_{\infty} \sum_{n}^{\infty} \dfrac{1}{n!} \right] \\ \text{and } \ \alpha_{\infty} = \dfrac{1}{2e^{-1}}, k_{\infty} = \dfrac{1}{2e} \end{cases} \tag{5.4}$$

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