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Derivative of biconfluent Heun's equation from some properties of hypergeometric function

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Abstract

The paper determines the solutions derived from the transformation of some confluent Heun's equation to hypergeometric equations by rational substitution. We show that the derivative of the original solution of the transformed confluent form of Heun's equation can be expressed in terms of another solution of the confluent form.

1.0 Introduction

The hypergeometric equation

$$z(1 - z)\frac{d^2y}{dz^2} + [c-(a+b+1)z]\frac{dy}{dz} - aby = 0 \qquad (1.1)$$

has precisely three regular singularities. This equation can be transform by some confluence process into the confluent form

$$\frac{d^2y}{dz^2} + P(z)\frac{dy}{dz} - Q(z)y=0 \qquad (1.2)$$

where $P(z) = (\frac{c-z}{z})$ and $Q(z) = \frac{a}{z}$.

In [1], it was shown that equation (1.2) could also be transformed into nontrivial confluent Heun's equation (CHE), double confluent Heun's equation (DHE), biconfluent Heun's equation (BHE) and triconfluent Heun's equation (THE) [2] Via a rational transformation if and only if this transformation is one of the six quadratic polynomials listed in Theorem below in section (1.1). These polynomials as well as the conditions imposed on the parameters for such a transformations to exist are tabulated explicitly. The case of general Heun's differential equation GHE [2] has been treated in [1]. By changing variable *x* to *t*, the general second order Heun's differential equation (GHE) can be transformed into the following four other multi-parameters equations CHE, DHE, BHE and THE, respectively [2]. We provide an answer to this in this paper. In deed, we provide that the derivative of the solution of **BHE** can be expressed in terms of another **BHE** solution. Adopting the same argument as in the case of GHE, we ask the question: When can a confluent hypergeometric equation be transformed into CHE, DHE, BHE and THE by a transformation z □= $R(t)$, where R □is a rational function of t?

1.1 Polynomial transformations

In this section we shall use the results in [2] to obtain a derived solution to the BHE via some polynomials transformation:

***Theorem* 1.1** [1, 2]

A hypergeometric equation (1.2) *can be transformed into a nontrivial CHE, DHE, BHE and THE by a rational transformation z* □= $R(t)$, *if and only if the singularities, where exists, form a harmonic system and* $R(t)$, *is one of the following quadratic polynomials*

$R= t^2$,□ $1 - t^2$,□ $(t - 1)^2$, □$2t$ □$- t^2$, □$(2t$ □$- 1)^2$, □$4t(1 - t)$,
□(2.3)

where the parameters α, β, γ, η□,δ □of CHE, DHE, BHE and THE depend explicitly on computable form of the two parameters a,□c □of (2.2).

It was discovered that only the BHE has a parameter relations via the polynomial $R(t) = t^2$.

2 0 Main result

2.1 Derived solutions obtained for BHE via polynomial transformation

Using the property of derivative of hypergeometric functions in the form

$$D_1F_1(a; c; z \,□= R(t)) = R'(t)\frac{a}{c}\,{}_1F_1\,(a + 1; c + 1; z□ = R(t))$$

we can deduce here the general relations between derivative of $B(\alpha, \beta, \gamma, \delta; z)$ and the hypergeometric function $F(a;c;z = R(t))$leading to another form of the **BHE** solution. For BHE the only polynomial transformation to hypergeometric function exists at $R(t) = t^2$, leading to the relations = $\alpha = 2c - 2$, $\beta = 0$, $\gamma = -4a + 2c$, $\delta = 0$.

Then we have the following relation obtained by pre-post application of the hypergeometric function:

$$DB(\alpha, 0, \gamma, 0; z = t^2) = \frac{(\alpha-\gamma+2)t}{\alpha+2}\,{}_1F_1\left(\frac{(\alpha-\gamma+2)}{4} + 1□; \frac{\alpha+2}{2} + 1; t^2\right)$$

$$= \frac{t(\alpha-\gamma+2)}{(\alpha+2)}B(\alpha+2, 0, \gamma-2, 0, ; z = t^2) \qquad (2.4)$$

since

$$\alpha = 2(\frac{\alpha+2}{2} + 1) - 2 \text{ and } \gamma = -4(\frac{(\alpha-\gamma+2)}{4} + 1) + 2(\frac{\alpha+2}{2} + 1)$$

and $B(\alpha, \beta, \gamma, \delta; z)$ is the analytic solution at z = 0.

3.0 Concluding remarks

The polynomial transformations led to the existence of new solution for only the BHE equation. Some other polynomial transformations are being investigated and shall be the content of the next paper in progress.

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