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Computational analysis of the Van der Pol equation using the Krylov-Bogolobov technique

[1]Y. M. Aiyesimi and T. Adeniyi,
Department of Mathematics and Computer Science
Federal University of Technology, Minna, Nigeria

Abstract

Analysis of non-linear equations has been a very difficult aspect of mathematics. This paper ventures into the specifics of Van Der Pol equation as related to the use of perturbation method of its analysis, finally employs the Krylov-Bogolobov techniques. Our purpose here is to describe the main features of the method of asymptotic expansion an example, that of Van Der Pol equation.

1.0 Introduction

Most problems encountered in nature take basic differential model either as lump parameter system or distributed parameter system, starting from engineering, astro-physics to operation research. One most encountered of such equation sis the van der pol equation which we would also find in quantum mechanics and celestial mechanics, etc. (Nayfeh A. H. 1973 [5]).

For this equation to have significant use in real application, their solution must converge to the real or exact solution. Approximation towards a defined point is known as asymptotic method (perturbation method). This is the approach adopted in this work.

The early days Mathematicians, most of whom were scientists in field like Physics therefore carved a niche for methods for analyzing systems with self-sustaining oscillation and this method is called PERTURBATION (Lugun N. I. et. al., 2000 [2]). Our important of such an equation is called the Van der Pol equation which was first encountered by Bathalza Van Der Pol (1927 [1]) who was a Radio engineer.

Mary L.C. and John L. (1940 [4]) worked on the solutions of the Van der Pol equation and discovered many of the phenomena that later became known as "chaos", which describe the output of a nonlinear radio amplifier when the input is a pure sine-wave. The whole development of radio in World War Two depended on high power amplifiers, and it was a matter of life and death to have amplifiers that did what they were supposed to do. The soldiers were plagued with amplifiers that misbehaved, and blamed the manufacturer for their erratic behavior. Cartwright and Littlewood discovered that the manufacturers were not to blame. The equation itself was to blame. They discovered that as you raise the gain of the amplifier, the solution of the equation become more and more irregular. At low power the solution has the same period as the input, but as the power increases you see solutions with double the period, and finally you have solutions that are not periodic at all.

McGillivray A.D. (1998 [3]) in his work "on the leading term of the inner asymptotic expansion of Van der Pol equation" proved rigorously that the leading term in the inner asymptotic expansion of the relaxation oscillation of Van der Pol's equation is correct. This was accomplished by constructing estimates which describes the difference between the asymptotic solution and the exact solution. These estimates were both rigorous and computable.

In most works that involves the solution using asymptotic expansion do not use time as the independent variable. Instead, one of the plane variables is expressed as an asymptotic expansion of the other plane variable. In this work we involve the time as independent variable and also considered the plane variables.

2.0 Problem formulation

The non-linear oscillators are in general governed by the differential equation, (Nayfeh A.H. et. Al. 1979 [6]);

$$\ddot{x} + n^2 x + \varepsilon f(x, \dot{x}) \qquad 2.1$$

[1]e-mail: yomi_aiyesimi2007@yahoo.co.uk

where ε determines the strength of non-linearity of the equation. For small values of ε the equation transforms to a weak, non-linear equation. Hence it approximates to

$$\ddot{x} + n^2 x \qquad 2.2$$

Equation (2.2) has a general solution

$$x(t) = a\sin(nt+\varphi) \qquad 2.3$$

Differentiating (2.3) we have

$$\dot{x}(t) = an\cos(nt+\varphi) \qquad 2.4$$

If the parameters a and φ now vary with t then we have;

$$\dot{x}(t) = an\cos(nt+\varphi) + \dot{a}\sin(nt+\varphi) \qquad 2.5$$

Hence, from (2.4) we have;

$$\dot{a}\sin(nt+\varphi) + a\dot{\varphi}\cos(nt+\varphi) = 0 \qquad 2.6$$

Differentiating (2.4) and substituting into (2.1) we obtain;

$$n\dot{a}\cos\psi - n^2 a\sin\psi = -\varepsilon f(a\sin\psi, an\cos\psi) \qquad 2.7$$

where $\psi = nt + \varphi$. From (2.6) an d(2.7) we have that;

$$\dot{a} = -\frac{\varepsilon}{n} f(a\sin\psi,\ an\cos\psi)\cos\psi \qquad 2.8$$

$$\dot{\varphi} = \frac{\varepsilon}{an} f(a\sin\psi,\ an\cos\psi)\sin\psi \qquad 2.9$$

As a first approximation to the values of $\dot{a}$ and $\dot{\varphi}$ we may replace the rhs of (2.8) and (2.9) with their respective average over a range of 2π for φ. This in effect gives the following:

$$\frac{da}{dt} = -\frac{\varepsilon}{2\pi n}\int_0^{2\pi} f(a\sin\psi,\ an\cos\psi)\cos\psi\, dt \qquad 2.10$$

$$\frac{d\varphi}{dt} = \frac{\varepsilon}{2\pi na}\int_0^{2\pi} f(a\sin\psi,\ an\cos\psi)\cos\psi\, dt \qquad 2.11$$

3.0 Krylov-Bogoliubov technique for the Van-der Pol equation

We now apply the method formulated above to solve the Van Der Pol differential equation;

$$\ddot{x} + n^2 x = \varepsilon(1-x^2)\dot{x} \qquad 3.1$$

Comparing equations (3.1) and (2.1) we have that;

$$f(a\sin\psi, an\cos\psi) = -(1-a^2\cos^2\psi)an\cos\psi \qquad 3.2$$

Recalling that for Van-Der Pol equation the parameter n= 1 we now have the result that;

$$\frac{da}{dt} = \frac{(4-a^2)a\varepsilon}{8} \qquad 3.3$$

and

$$\frac{d\varphi}{dt} = 0 \qquad 3.4$$

The results of (3.3) and (3.4) are respectively;

$$a(t) = \frac{2\sqrt{Ar^{\varepsilon t/2}}}{1+Ae^{\varepsilon t}} \text{ and } \varphi = cons\ \tan t \qquad 3.5$$

Thus for a weak non-linear oscillator we have the result;

$$x(t)=\frac{2\sqrt{Ar^{\varepsilon t/2}}}{1+Ae^{\varepsilon t}}\sin(t+\varphi)$$

4.0 Numerical simulation

In what follows we have assumed a unit initial displacement and a phase constant of $\frac{\pi}{2}$ to study the effect of changing values of the strength of the non linear parameter as it affects the amplitude of vibration.

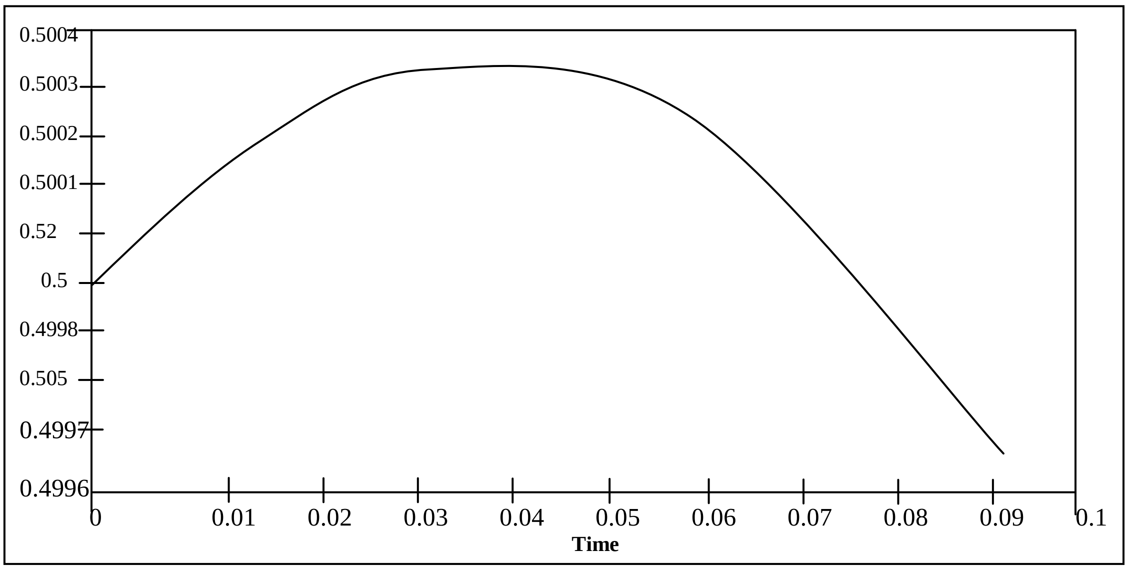

Figure 4.1: General Displacement of a non-linear system with ε = 0.1

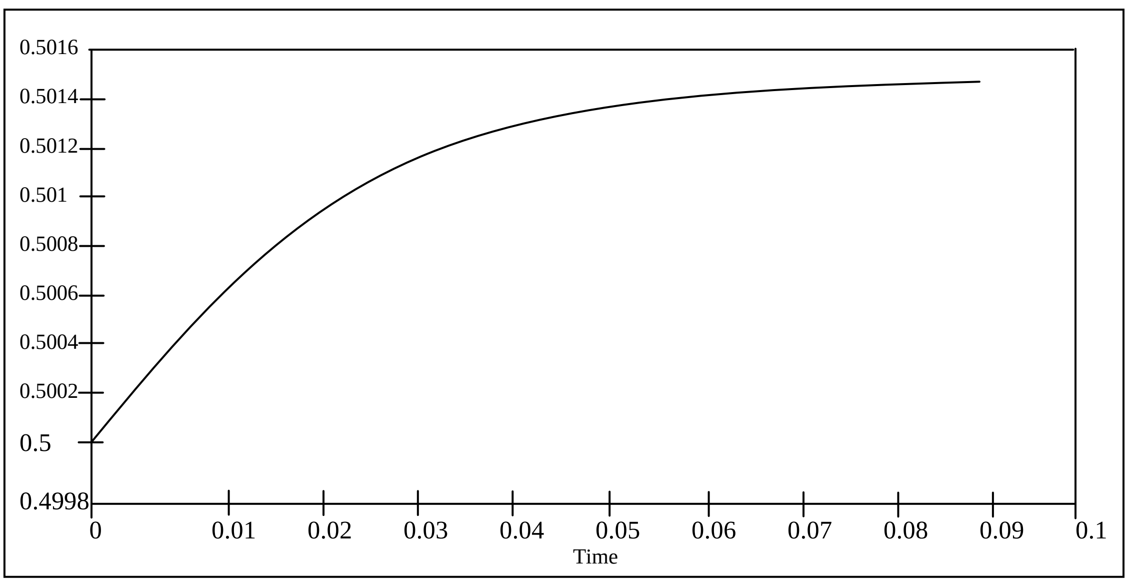

Figure 4.2: General Displacement of a non-linear system with ε = 0.2

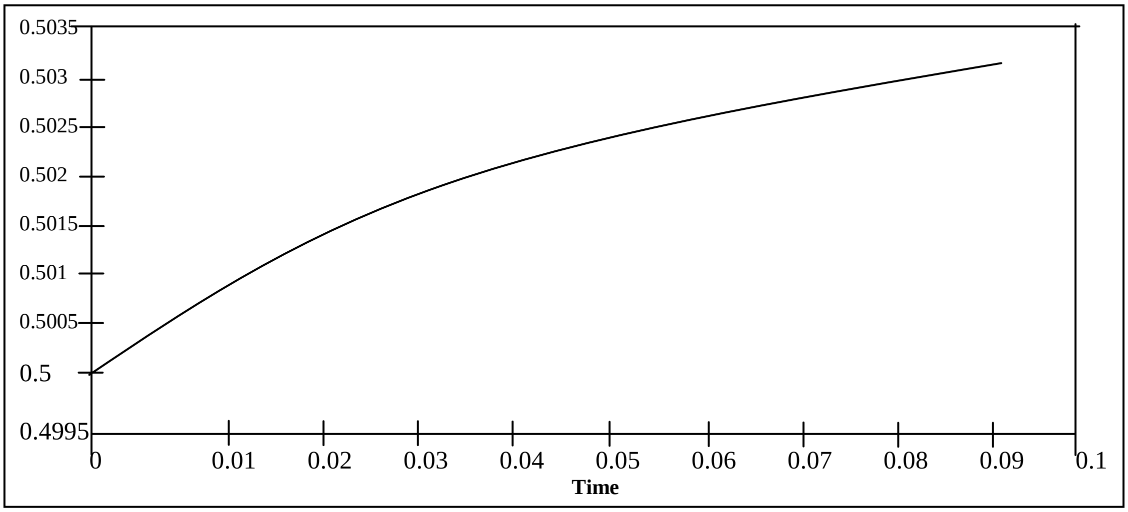

Figure 4.3: General Displacement of a non-linear system with $\varepsilon = 0.3$

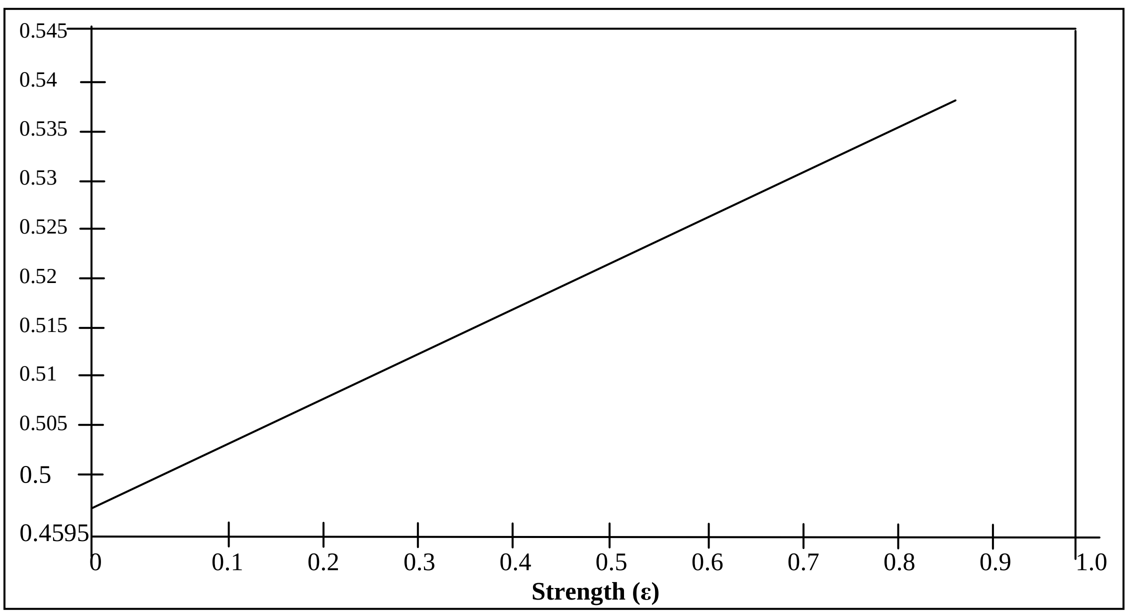

Figure 4.4: General Displacement of a non-linear system with $t = 0.01$

5.0 Conclusion and discussion of results

The graphs of oscillation against time were plotted with different values of (ε) strength. It is shown that in relaxation oscillation i.e. in the Van Der Pol's equation if the damping co-efficient is negative then system is unstable for small displacement where for large displacement the damping coefficient becomes positive. This small oscillation tends to diminish and a stable result.

The Van Der Pol's oscillation can be used in studying and understanding realization oscillation. For large displacement, the damping coefficient becomes negative. Finally, the analysis shows that the Van Der Pol's equation has unusual limit cycle.

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