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Some results on the generalized projection and asymptotic operators

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1.0 Introduction

Let X be a normed linear space with dual X^*. We denote by J the normalized duality mapping form X to 2^{X^*} defined by

$$Jx = \{ f \in X^* : < x, f > = \|x\|^2 = \|f\|^2 \},$$

Where <.,.> denotes the generalized duality pairing. It is well-known that if X^* is strictly convex then J is single-valued and if X^* is uniformly convex then J is uniformly continuous on bounded subsets of X, (see e.g. [8]). We shall denote the single-valued duality map by j. Let E be a Banach space. We recall the definition of the generalized projection operator recently introduced by ALbr-Delabriere (see e.g. [1, 2]) as follows: with E as above, $K \subseteq E$, closed, convex subset of E. Let

$$\Pi_K : E \to K$$

defined by $Pi_K x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$V(x,\bar{x}) = \inf V(x,\zeta)$$

and

$$V(x,\zeta) = \|x\|^2 - 2 < x, j(\zeta) > + \|\zeta\|^2,\ j(\zeta) \in J(\zeta).$$

It is shown [2] that $V(x,\zeta)$ can be considered not only as square of distance between points x and ζ but also as a Lyapunov function withrespect to ζ with fixed x. In Hilbert (and only in Hilbert) spaces, generalized projection coincides with the usual metric project. Some of the properties of Π_K are shown in [2].

The Banach space E is said to be uniformly convex if $\delta_E(\varepsilon) \geq 0\, \forall \varepsilon \in (0,2]$, where the function $\delta_E : (0,2] \to [0,1]$ is defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1,\ \|x, y\| \geq \varepsilon\right\}$$

The modulus of smoothness of E is the function ρ_E defined by

$$\rho_E(\tau) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau, \tau > 0\right\}.$$

E is said to be uniformly smooth if $\lim_{\tau \to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$.

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A map *T* with domain *D*(*T*) and range *R*(*T*) in *E* is called weakly contractive (see e.g. [1, 2]) if there exists a continuous and nondecreasing function $\phi:[0,\infty)\to\Re^{+}$ such that ϕ is positive $\Re^{+}-0, \phi(0)=0, \lim_{t\to\infty}\phi(t)=\infty$ and for $x, y\in D(T)$, there exits $j(x-y)\in J(x-y)$ such that

$$\|Tx-Ty\| \le \|x-y\|-\phi(\|x-y\|).$$

T is called strongly suppressive on a closed convex set $G\subseteq E$ if there exists $c\in(0,1)$ such that for all $x, y,\in G$,

$$V(Tx,Ty) \le cV(x,y)$$

T is called weakly suppressive of class $C_{\phi(t)}$ on a closed convex set $G\subseteq B$ if there exists a continuous and none decreasing function ϕ(t) defined on $\Re^{+}$ such that ϕ is positive on $\Re^{+}-0, \phi(0)=0, \lim_{t\to\infty}\phi(t)=\infty$ and $\forall x,\ y\in G$,

$$V(Tx,Ty) \le V(x,y)-\phi(V(x,y))$$

T is called nonextensive on a closed convex set $G\subseteq B$ if for all $x, y,\in G$,

$$V(Tx,Ty) \le V(x,y).$$

It is clear that every weakly suppressive mapping is nonextensive and every strongly suppressive mapping is weakly suppressive by setting $\phi(t)=(1-c)t$. In Hilbert spaces, strongly suppressive operators are weakly contractive and nonextensive mappings are nonexpansive and vice versa.

In [1,2], Alber-Guerre proved convergence results with the generalized projection for the suppressive and contractive-type operators. The author (with C.E. Chidume and H. Zegeye) also generalized some of these results in different directions, see [4].

Let *K* be a nonempty subset of a normed space *E*. A mapping $T:K\to K$ is called asumptotically nonextensive (see e.g. [6] if there exists a sequence $\{k_n\}, k_n\ge 1$, such that $\lim_{n\to\infty}k_n=1$, and

$$\|T^n x-T^n y\| \le k_n\|x-y\|$$

for each $x, y\in K$ and for each integer $n\ge 1$. *T* is called asymptotically pseudocontractive (see e.g. [6] if there exists a sequence $\{k_n\}, k_n\ge 1, \lim_{n\to\infty}k_n=1$ such that

$$\langle T^n x-T^n y,\ j(x-y)\rangle \le k_n\|x-y\|^2,$$

for each, $x, y\in K$. The asymptotic operators were first introduced by Goebel and Kirk, while the contractive and suppressive mappings were first introduced by ALber-Guerre (see for e.g. [1, 2, 6]. These classes of mappings have been studied by various authors (see e.g. . [1, 2, 3, 4]. Motivated by Alber-Guerre and Goebel-Kirk, we now introduce the class of asymptotically weakly suppressive maps.

***Definition* 1.1**

The map $T:G\to E, G\subseteq E$, will be called asymptotically weakly suppressive of class $C_{\phi(t)}$ if there exists a nondecreasing continuous map $\phi:\Re^{+}\to\Re^{+}$ such that $\phi(0)=0,\ \phi(t)\ge t,\ \lim_{t\to\infty}=\infty$ and a sequence $\{k_n\}$ such that $k_n\ge 1,\ \lim_{n\to\infty}k_n=1$ and $\forall\ x, y\in G$, there exists $j(x-y)\in j(x-y)$ such that

$$V(T^n x-T^n y) \le k_n V(x,y)-\phi(V(x,y)).$$

In this paper, we obtain some new results with the generalized projection operator involving the Ishikawa-type iteration and some contractive and suppressive-type mappings. We study the asumptotic operators involving the weakly suppressive mappings with the help of the generalized projection.

2.0 Main result

***Lemma* 2.1**

Let $\{\rho_i)$ *be a sequence of nonnegative numbers and* $\{k_i\}$ *a sequence of positive numbers such that* $k_i \ge 1, \lim_{i\to\infty} k_i = 1$. *Let the recursive inequality* $\rho_{n+1} \le k_n\rho_n - \phi(\rho_n), n = 1,2,\ldots$ *be given where*

$\phi(t)$ is a continuous and non decreasing function from $\Re^+$ *to* $\Re^+$ *such that it is positive on* $\Re^+ - 0, \phi(0) = 0, \phi(t) \ge t, \lim_{t\to\infty}\phi(t) = \infty$. *Then* $\rho_n \to 0$ *as* $n \to \infty$.

Proof.

Let $\lim\inf \rho_n = \alpha$. We claim that $\alpha = 0$. For if not, then there is N_1 such that $\forall n \ge N_1$,

$$\lim\inf \rho_n = \alpha > 0 \qquad (2.1)$$

Thus there is a subsequence $\{\rho_{nj})$ such that, $\rho_{nj+1} > k_{nj}\rho_{nj} - \phi(\rho_{nj})$,

that is

$$k_{nj}\rho_{nj} - \rho_{nj+1} < \phi(\rho_{nj}) \qquad (2.2)$$

Since ρ_{nj} and ρ_{nj+1} have same limit, we obtain from 2.1, $\sum_{j=N1}^{\infty}(K_{Nj} - 1)\rho_{nj} < \sum_{n\ge N1}\phi(\alpha) = \infty,$

that is,

$$\sum_{j=N1}^{\infty}(K_{Nj} - 1)\rho_{nj} < \infty, \qquad (2.3)$$

Since k_i is bounded, say by M, then we get from 2.2 that $\sum_{j=N1}^{\infty}\rho_{nj} < \infty$. Thus the sequence $\{\rho_{nj}\} \to 0$ as $n \to \infty$. This contradicts 1. Thus $\alpha = 0$.

Claim $\rho_n \to 0$ as $n \to \infty$.

For if not, there is a subsequence $\rho_{nj}*$ such that $\rho_{nj}*$ does not tend to zero. Since $\rho_{nj}\to 0$, given $\varepsilon > 0$, there is N_2 such that $\rho_{nj} < \frac{\varepsilon}{2} \forall n > N_2$. We show that for any $m \in N \cup \{0\}$, $\rho_{nj+m} < \frac{\varepsilon}{2}$. For k= 0, there is trivial. Assume true for any m, we show that it is true for m +1, i.e. $\rho_{nj+m+1} < \frac{\varepsilon}{2}$. Assume for contradiction that $\rho_{nj+m+1} < \frac{\varepsilon}{2}$, then

$$\tfrac{\varepsilon}{2} < \rho_{nj+m+1} \le k_{nj+m}\rho_{nj+m} - \phi(\rho_{nj+m+1}) \le \left(1 + \frac{\varepsilon}{2}\right)\rho_{nj+m} - \phi\left(\frac{\varepsilon}{2}\right)$$

i.e.

$$\frac{\varepsilon}{2} < \rho_{nj+m+1} \le \rho_{nj+m} + \frac{\varepsilon}{2}\rho_{nj+m} - \frac{\varepsilon}{2} < \frac{\varepsilon^2}{4},$$

i.e.

$$\frac{\varepsilon}{2} < \rho_{nj+m+1} < \frac{\varepsilon^2}{4},$$

a contradiction. Hence $\rho_n \to 0$ as $n \to \infty$.

***Theorem* 2.2**

Let E be a uniformly convex and uniformly smooth Banach space. $T : G \to E$ be an asymptotically weakly suppressive map on a closed convex set $G \subseteq E$. Then the sequence $\{x_n\}$ generated by

$$x_{n+1} = \pi_G T^n x_n$$

converges strongly to the fixed point of T.

Proof

$$V(x_{n+1},x^*)=V(\pi_K T^n x_n,\pi_k Tx^*)\le V(T^n x_n,Tx^*)\le k_n V(x_n,x^*)-\phi(V(x_n,x^*))$$

Using Lemma 2.1, we have that $\rho_n \to 0$ as $n\to\infty$. , i.e. $V(x_n,x^*)\to 0$ as $n\to\infty$. We now invoke Alber-Guerre ([1], pg. 24) to get $\lim_{n\to\infty}\|x_n - x^*\|=0$, i.e. $x_n \to x^*$ as $n\to\infty$, concluding the proof of the theorem.

***Theorem* 2.3**

Let E be a uniformly convex and unfiromly smooth Banach space and $T:G\to E$ be an asymptotically nonextensive operator. Then the sequence $x_{n+1}=\pi_G T^n x_n$ converges to the fixed point of T,

Proof

$$V(x_{n+1},x^*)=V(\pi_G T^n x_n,\pi_G T^n x^*)\| \le V(T^n,T^n x^*)\le k_n V(x_n,x^*)\le k_1 V(x_1,x^*).$$

We now invoke Aber-Guerre ([1], pg. 25) to conclude that $x_n \to x^*\ n\to\infty$. We now state and prove the following theorem involving the Ishikawa iteration scheme.

***Theorem* 2.4**.

Let H be a Hilbert space and $G\subseteq H$ be a weakly contractive map from $G\to H$ of the class $C_{\phi(t)}$ and $x^*\in G$ its fixed point. Then the iterative sequence defined by

$$x_{n+1}=P_G((1-\alpha_n)x_n+\alpha_n Ty_n),\ y_n=(1-\beta_n)x_n+\beta_n Tx_n)$$

and $(1-\alpha_n)+\alpha_n(1-\beta_n)+\alpha_n\beta_n<1$, converges strongly to the fixed point of *T*.

Proof

$$\|x_{n+1}-x^*\|=\|P_G((1-\alpha_n)x_n+\alpha_n Ty_n-P_G x^*\|$$
$$\le \|(1-\alpha_n)x_n+\alpha_n Ty_n-x^*\|=\|x_n-x^*+\alpha_n(Ty_n-x_n)\|$$
$$=\|(1-\alpha_n)(x_n-x^*)+\alpha_n(Ty_n-Tx^*)\|\le \|(1-\alpha_n)\|x_n-x^*\|+\alpha_n\|Ty_n-Tx^*\|$$
$$\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|T(1-\beta_n)x_n+\beta_n Tx_n]-Tx^*\|$$
$$=(1-\alpha_n)\|x_n-x^*\|+\alpha_n\|[(1-\beta_n)x_n+\beta_n Tx_n-x^*\|-\alpha_n\phi\{[(1-\beta_n)x_n+\beta_n Tx_n-x^*]\}$$
$$=(1-\alpha_n)\|x_n-x^*\|+\alpha_n\{(1-\beta_n)\|x_n-x^*\|+\beta_n\|Tx_n-Tx^*)\|\}-\alpha_n\phi(.)$$
$$=(1-\alpha_n)\|x_n-x^*\|+\alpha_n(1-\beta_n)\|x_n-x^*\|+\beta_n\|x_n-x^*\|-\beta_n\phi(\|x_n-x^*\|)-\alpha_n\phi(.)$$
$$=(1-\alpha_n)\|x_n-x^*\|+\alpha_n(1-\beta_n)\|x_n-x^*\|+\alpha_n\beta_n\|x_n-x^*\|-\alpha_n\beta_n\phi(\|x_n-x^*\|)-\alpha_n\phi(.)$$
$$=[(1-\alpha_n)+\alpha_n(1-\beta_n)+\alpha_n\beta_n]\|x_n-x^*\|-\alpha_n\beta_n\phi(\|x_n-x^*\|)-\alpha_n\phi(.)$$
$$\le [(1-\alpha_n)+\alpha_n(1-\beta_n)+\alpha_n\beta_n]\|x_n-x^*\|-\alpha_n\beta_n\phi(\|x_n-x^*\|)$$

Since $(1-\alpha_n)+\alpha_n(1-\beta_n)+\alpha_n\beta_n<1$, then last inequality satisfies

$$\|x_{n+1}-x^*\|\le \|x_n-x^*\|-\gamma_n\phi(\|x_n-x^*\|)$$

with $\gamma_n<1$. With $\rho_n=\|x_n-x^*\|$, the last inequality reduces to $\rho_{n+1}\le \rho_n-\gamma_n\phi(\rho_n)$. We now invoke ALber-Guerre ([1], pp.33) to conclude that $\rho_n\to 0$, i.e. $x_n\to x^*$ as $n\to\infty$.

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