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Number of fixed point free element in the subgroup of orientation reversing mappings

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Abstract

Let ORD_n be the subgroup of all orientation reversing bijective mappings of n-element set. It is shown that for m even: if $m = 2k + 1$, then $\rho_m \in ORD_n$ has no fixed point. And if m is odd there are exactly $n/2$ derangements.

Keywords: Permutation, derangement, symmetries, full transformation, partial transformation, partial one-one transformation.

1.0 Introduction and preliminaries

Let X_n denote the set $\{1,2,\ldots n\}$ considered with standard ordering and let T_n, P_n and O_n, be the full transformation semigroup, the partial transformation semigroup and the submonoid of T_n consisting of all order preserving mappings of X_n, respectively. Another closely related algebraic structure to O_n and P_n are S_n and D_n the symmetric and dihedral groups on the set X_n, respectively.

Catarino and Higgins [12] introduced a new subsemigroup of X_n containing O_n which is denoted by OP_n and its elements are called orientation preserving mappings. They also introduced a semigroup $P_n = OP_n \cup OR_n$ where OR_n denotes the collection of all orientation reversing mappings. Fernandes [14] studied the monoid of orientation preserving partial transformations of a finite chain, concentrating in particular on partial transformations which are injective. Bashir [7] considered the subgroup of orientation preserving bijective mappings. Here, we consider the subgroup of orientation reversing bijective mappings. In particular, we pay attention to a subgroup of the Dihedral group D_n of the order $2n$ defined as

$$D_n = \{x, y \mid x^n = 1,\ y^2 = 1 \quad xy = x^{-1}y\}.$$

Combinatorial properties of T_n and S_n and some of their semigroups and subgroups respectively, have been studied over a long period and many interesting and delightful results have emerged (see for example [11], [8], [9]). Recently, inspired by the works of Bashir and Umar [6], Bashir and Umar [7], Bashir [8] has shown that for n-odd there are n-even derangements, and for n-even there are $\frac{n}{2}$ even and $\frac{n}{2}$ odd derangements, respectively, in OPD_n (the subgroup of all orientation preserving bijective mappings of n-element set.).

At the end of this introductory section we gather some known results that we shall need in later sections. In section 2 we prove some results which we will need in the proof of the main results. Finally, in section 3, we obtain the number of fixed point free elements for n-even (odd) in the subgroup.

Throughout the remaining of this paper, $m, n,$ and $k \in$ N, $n > m > k$, $0 \le k \le \frac{m-1}{2}$, and $0 \le m \le n - 2$. If $m = 2k + 1$ then $0 \le k \le \frac{m+1}{2}$ and $0 \le m \le n-2.$ Let $\rho \in ORD_n$, we say ρ is an orientation-

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reversing bijective mapping on X_n if the sequence $(1\rho,2\rho,\ldots,n\rho)$ is anti-cyclic, the collection of all orientation-reversing bijective mappings on X_n is denoted by ORD_n.

We define a reflection $\rho_m \in ORD_n$ by $i \to n - i - m + 1 (i \in X_n)$ with $\rho = \rho_0 = i \to n + 1 - i$ such that $(1\rho,2\rho,\ldots,n\rho) = (n,n-1,\ldots,1)$ and is anti-cyclic. Thus, for every $\alpha^m \in OPD_n$ there exist an equivalence $\alpha^m\rho$ in ORD_n. That is there exist an isomorphism between the subgroups OPD_n and ORD_n

A map $\alpha:X_n \to X_n$ is order decreasing if $x\alpha \le x$, for all x in X_n. If $x \le y \Rightarrow x\alpha \le y\alpha$, then α is said to be order preserving for all x,y in X_n. Let $A = (a_1,a_2,\ldots,a_s)$ be a finite sequence from the chain X_n. We say that A is cyclic or has clockwise orientation if there exist not more than one subscript i such that $a_i > a_{i+1}$ where a_{s+1} denotes a_1. We say that $A = (a_1,a_2,\ldots,a_s)$ is anti-cyclic or has anticlockwise orientation if there exists no more than one subscript i such that $a_i < a_{i+1}$. Note that a sequence A is cyclic if and only if A is empty or there exist $i \in \{0,1,\ldots,s-1\}$ such that $a_{i+1} \le a_{i+2} \le \ldots \le a_s \le a_1 \le \ldots \le a_i$. i is unique unless the sequence is a constant.

Recall from [1] that an even permutation is a permutation which can be expressed as a product of an even number of cycles of even length and/or a product of any number of cycles of odd length. A permutation that is not even is called odd. The set of even permutations of X_n, called the alternating group is usually denoted by A_n. Recall also that, a derangement σ is a permutation such that $\sigma(x) \ne x$ that is, a permutation without fixed points.

***Result* 1.1**

Let A be any cyclic (anti-cyclic) sequence. Then A is anti-cyclic (cyclic) if and only if A has no more than two distinct values. If $A=(a_1,a_2,\ldots,a_t)$ is any sequence then we denote by A^τ sequence $(a_t,a_{t-1},\ldots,a_1)$, called the reversed sequence of A.

***Result* 1.2**

Let $A=(a_1,a_2,\ldots,a_t)$ be any sequence from X_n. Then A is cyclic (anti-cyclic) if and only if A^τ is anti-cyclic (cyclic).

***Result* 1.3**

If $(a_1,a_2,\ldots,a_t)$ is cyclic (anti-cyclic) then so is

(a) the sequence. $(a_{i_1},a_{i_2},\ldots,a_{i_r})$ $(i_1 < i_2 < \ldots < i_r)$ and (b) the sequence. $(a_j,a_{j+1},\ldots a_t,a_1,\ldots,a_{j-1})$, for all $1 \le j \le t$.

2.0 Subgroup of orientation reversing mapping

We will use the following results, adapted to the subgroup of Orientation reversing mapping case, which is easily proved.

We list some known results which may be found in [12], [14] that we shall need later.

***Proposition* 2.1**

Any restriction of a member of ORD_n is also a member of ORD_n.

***Proposition* 2.2**

Let $\alpha \in ORD_n$ and let $(a_1 \ldots a_m)$, $m \ge 1$ be any cyclic sequence of members of X_n, then the sequence $(a_1\alpha \ldots a_m\alpha)$ is also cyclic. Similarly $((a_1\alpha)\alpha \ldots (\alpha_m\alpha)\alpha)$ is cyclic.

***Proposition* 2.3 [11, Lemma 4.8]**

Let $\alpha \in ORD_n$.Then the digraph of α cannot have a non-trivial cycle and a fixed point.

***Proposition* 2.4 [11, Lemma 4.9]**

Let $\alpha \in ORD_n$. Then the digraph of α cannot have two cycles of different length.

3.0 Number of fixed point free element

***Lemma* 3.1**

If n is even, for $m = 2k$, ρ_m has no fixed point.

Proof

The assertion may be proof by induction on m there are several cases to examine. First recall that for all $\rho_m \in ORD_n,\ \rho_m = \alpha^m \rho = \prod_{i=1}^{\frac{n+1}{2}} (i, n - m - i + 1)$. For n even we consider two cases of m, $m = 2k$.

Case I. ***m*** **= 2*k* and *n*-even**.

Consider $m = 0$ and 2. $\rho = \alpha^0 \rho = \prod_{i=1}^{n} (i,\ \ n - i + 1)$. If i is a fixed point, then $0 \to i = \frac{n+1}{2}$

It does not exist in N (set of natural numbers) or ρ.

If $i = \frac{n+1}{2}$ is a fixed point, then it is clear that n - 1 is odd, since n is even, it implies that $i = \frac{n+1}{2}$ does not exist in N (set of natural numbers) or ρ. Hence or otherwise, if $i = \frac{n+1}{2}$ is a fixed point, then, $0 \to i = n \neq 1$, $\rho = \rho_0 = (1\ \ n)(2\ \ n-1)\Lambda \left(\frac{n-2}{2}\ \ \frac{n+4}{2}\right)\left(\frac{n}{2}\ \ n+1\right)$. If $m = 2$, $\rho = \alpha^2\rho = (1, n - i - 1)$ If i is a fixed point, then $i = \frac{n-1}{2}$, it does not exist in N (set of natural numbers) or ρ since, by similar argument as in $m = 0$, n is an even natural number; n - 1 is odd $\frac{n-1}{2} \notin \mathrm{N}$. If we assume that $\frac{n-1}{2} = n-1$ as in other case, an odd number, that $i = n - 1$ is a fixed point, it implies that $i \to n - (n - 1) - 1 \neq n - 1$. Hence ρ_2 doesn't have a fixed point.

$$\rho_2 = (1\ \ n-2)(2\ \ n-3)\Lambda \left(\frac{n-4}{2}\ \ \frac{n+2}{2}\right)\left(\frac{n-2}{2}\ \ \frac{n}{2}\right) \Lambda \ (n\ \ n-1)$$

Case II.

Assume that the result is true for $m = 2k$, $\rho_{2k} = (i, n - i - 2k + 1)$. If $i = \frac{1}{2}(n - 2k + 1)$ is a fixed point, then it is clear that $n - (2k - 1)$ is odd, since n is even and $2k$ - 1 is odd. It implies that $\frac{n-(2k-1)}{2}$ does not exist in N (set of natural numbers) or ρ.. Hence or otherwise, if we assume that $i = \frac{n-(2k-1)}{2} = n-(2k-1)$, is a fixed point, then $2k \to i = n - (n - (2k - 1)) - 2k + 1 \neq n - (2k - 1)$.

$$\rho_{2k} = (1,\ \ n-2k)(2\ \ n-2k-1)\Lambda \left(\frac{n-2k-1}{2}\ \ \frac{n-2k+3}{2}\right)\left(\frac{n-2k}{2}\ \ \frac{n-2k+2}{2}\right)\Lambda$$
$$(n-1\ \ n-2k+2)(n\ \ \ n-2k+1)$$

Case III. $m = 2(k+1)$, the next even natural number after $2k$,

$$\rho_m = \alpha^{2(k+1)} = (i,\ n - i - 2k - 1)$$

$$\rho_m = (1\ \ n-2(k+1))(2\ \ n-2(k+1)-1)\Lambda \left(\frac{n-2(k+1)-1}{2}\ \ \frac{n-2(k+1)+3}{2}\right)$$
$$\left(\frac{n-2(k+1)}{2}\ \ \frac{n-2(k+1)+2}{2}\right)\ \Lambda\ (n-1\ \ \ n-2(k+1)+2)(n\ \ n-2(k+1)+1)$$

$$i = \frac{n-2(k+1)+1}{2} \Rightarrow i = \frac{n-m+1}{2},\ \ m = 2(k+1)$$

does not exist as a fixed point in ρ_m, by a similar argument as in the cases I and II above $n=m+1$ is odd, which implies that $\frac{n-m+1}{2}$ does not exist in N (or $\rho = 2(k+1)$). Hence or otherwise, if $i=\frac{n-m+1}{2}=n-m+1$ then $2k+1 \rightarrow i = n-(n-m+1) - 2k-1 \neq n-m+1$ implies that $n-2(k+1)+1$ is not a fixed point. The induction process proves that the result is true for any value of $m = 2k$, n-even.

***Lemma* 3.2**

If n is even and m-odd for every ρ_m Є ORD_n there are exactly n/2 *derangements.*

Proof

If $n = 4k$ or $(4k + 2)$ is even, it is clear that there are $n/2$ even numbers of *m's* in *n*, and $n/2$ odd numbers of *m's* in *n* If $m = 2k + 1$ there are $n/2$ odd *m's*' in *n* It implies that we $n/2$ permutations with two fixed points in *n* by Lemma 3.1. Similarly, by the same argument and Lemma 3.2 there are $n/2$ derangements in n.

Table 3.1: *n*-even permutations of *k* field points.

n \ *k*	0	1	2	3	4	5	6	7	8	9	$\Sigma e(n,k)$
0	1										1
1	0	1									1
2	0	0	1								1
3	2	0	0	1							3
4	3	0	0	0	1						4
5	4	5	0	0	0	1					10
6	2	0	3	0	0	0	1				6
7	6	0	0	0	0	0	0	1			7
8	7	0	0	0	0	0	0	0	1		8
9	8	9	0	0	0	0	0	0	0	1	18

Table 3.2: *n*-odd permutations of *k* field points.

n \ *k*	0	1	2	3	4	5	6	7	8	9	$\Sigma e'(n,k)$
0	0										0
1	0	0									0
2	0	0	0								0
3	0	3	0	0							3
4	2	0	2	0	0						4
5	0	0	0	0	0	0					0
6	6	0	0	0	0	0	0				6
7	0	7	0	0	0	0	0	0			7
8	4	0	4	0	0	0	0	0	0		8
9	0	0	0	0	0	0	0	0	0	0	0

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