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On Rouche's theorem as a criterion for irreducibility of polynomial maps.

K. B. Yuguda
Department of Mathematics and Computer Science,
Nigerian Defence Academy, Kaduna.

e-mail: kyuguda@yahoo.com, Phone 08033691107

Abstract

In this paper, a proof of Rouche's Theorem for irreducible polynomials is presented using the unit circle as our closed curve under consideration. It will be noted that, use is made of the Fundamental Theorem of Algebra just as it is well known that the Fundamental Theorem of Algebra can also be proved using Rouche's Theorem. Thus, one may rightly view this paper as establishing that Rouche's Theorem for polynomial maps is equivalent to the Fundamental Theorem of Algebra. (i.e. that each is easily derivable from the other.)

Keywords: *argument, irreducible, multiplicity of zeros, poles, unit circle.*

1.0 Introduction

The well known Rouche's theorem (cf. Conway [1]) often allows one to determine the Relative location of the zeros or poles of a Meromorphic function $f(z)$ by comparing $f(z)$ to another Meromorphic function whose zeros and poles are already known. In the case of polynomial maps this is particularly useful in deducing the irreducibility of a given Polynomial, $f(z)$. Thus, for example, in this manner, one can deduce Peron's theorem [4].

If $f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$, has integer coefficients with $a_0 \neq 0$ and $|a_{n-1}| \geq 1 + |a_{n-2}| + \ldots + |a_0|$ then $f(z)$is irreducible over the integers.

In this paper, therefore, we present a proof of Rouche's Theorem as a criterion for Irreducibility of Polynomial Maps. Without loss of generality, we shall take as our point of departure the simplest case where the closed curve under consideration is the unit circle. It will be clear from the proof that more general curves could be used. Before proceeding however, we note that the proof given here makes use of the Fundamental Theorem of Algebra. On the other hand we can prove the Fundamental Theorem of Algebra using Rouche's Theorem. Thus, one may rightly view this paper as establishing Rouche's Theorem for Polynomial maps is equivalent to the Fundamental Theorem of Algebra (i. e., that each is easily derivable from the other).

We add that proofs of the Fundamental Theorem of Algebra which do not require Rouche's Theorem can be found in Dickson [2], Grove [3] or Rudin [5] , and it is an easy consequence of Liouville's Theorem.

2.0 Statement and proof of result

We state here the identity which is central to our work; Rouche's Theorem and prove in the sequel its role as a criterion for irreducibility.

***Theorem* 2.1**

Let $f(z)$ and $g(z)$ be polynomials with complex coefficients and $C = \{z:|z| = 1\}$. If the strict inequality $| f(z) + g(z) | < | f(z) + | g(z) |$ holds at each point on the circle C, then $f(z)$ and $g(z)$must have the same total number of zeros (counting multiplicity) strictly inside C.

We shall require the following lemma which we proceed to give.

***Lemma* 2.1**

The roles of $f(z)$ and $g(z)$ in the above theorem is symmetric.

***Proof of theorem* 2.1**

Let k and t be the number of zeros of $f(z)$ and $g(z)$ inside C, respectively. By Lemma 2.1, it suffices to show that $k \leq t$. Assume on the contrary that $k > t$. We will obtain, a contradiction, by showing that there is $\varphi \in [0,2\pi]$, such that

$$| f(e^{i\varphi}) + g(e^{i\varphi}) | = | f(e^{i\varphi}) | + | g(e^{i\varphi}) | \tag{2.1}$$

By the triangle inequality and following closely the work of Yuguda [6], it suffices to show that there is $\varphi \in [0,2\pi]$ such that arg($f(e^{i\varphi})$) = arg($g(e^{i\varphi})$), for $z, \alpha \in C$, $|\alpha| \neq 1$, we take $\arg\ z \in [-\pi, \pi]$, define $w(\alpha,0)$ = arg(1 – α) and for θ Є R – {0}, choose $s = s(\theta)$ an integer so that , $w(\alpha,0) = \arg(e^{i\varphi} - \alpha) + 2s\pi$ is continuous on R.

Geometrically speaking, $w(\alpha,\theta) - w(\alpha,0)$ is a continuous representation of the angle with vertex at α and rays emanating through 1 and $e^{i\varphi}$. Thus,

$$w(\alpha;2\pi) - w(\alpha;0) = \begin{cases} 2\pi & , if\ \alpha \in C \\ 0 & , otherwise \end{cases} \tag{2.2}$$

Let a and b be the leading coefficients of $f(z)$ and $g(z)$ respectively and $\alpha_i, i = 1,2,...,n$ and β_j = 1,2,…,m be their zeros (the zeros appearing as many times as their multiplicities). The inequality in the hypothesis of the theorem implies that each $|\alpha_i| \neq 1|$ and each $|\beta_j| \neq 1$. Define

$$F(\theta) = \arg(a) + \sum_{i=1}^{n} w(\alpha_i;\theta) + 2u\pi \tag{2.3}$$

where u is an integer chosen so that $F(0) \in (-2\pi,0]$. Similarly, define

$$G(\theta) = \arg(b) + \sum_{j=1}^{m} w(\beta_j;\theta) + 2v\pi \tag{2.4}$$

where v is an integer chosen such that $G(0) \in [F(0), F(0) + 2\pi)$. Thus $F(0)$ and $G(0)$ are continuous on R. Also, since

$$\arg(f(e^{i\theta})) = \arg(a \prod_{i=1}^{n} (e^{i\theta} - \alpha_i)) \equiv \arg(a) + \sum_{i=1}^{n} \arg(e^{i\theta} - \alpha_i) (\text{mod } 2\pi),$$

We deduce from (2.2) that,

$$F(\theta) = \arg(f(e^{i\theta}))(\text{mod } 2\pi) \tag{2.5}$$

Similarly,
$$G(\theta) = \arg(g(e^{i\theta}))(\text{mod } 2\pi) \tag{2.6}$$

Furthermore, by equation (2.2), we have $F(2\pi) = F(0) + 2k\pi$ and $G(2\pi) = G(0) + 2t\pi$, respectively.

Let $H(\theta) = F(\theta) - G(\theta)$. Then, $H(0) = F(0) - G(0) \leq 0$, and $H(2\pi) = F(2\pi) - G(2\pi) = 2(k-t)\pi + F(0) - G(0) > 0$. Hence by the Intermediate Value Theorem, there is a φ Є $[0,2\pi]$ for which $H(\varphi) = 0$. By (2.5) and (2.6), we now obtain that, arg($f(e^{i\varphi})$) = arg($g(e^{i\varphi})$), this completes the proof.

References

[1] J. B. Conway, Functions of one Complex Variable .2nd edition. Springer Verlag. New York 1978 p125

[2] L.E. Dickson, New First Course in the Theory of Equations. John Wiley and Sons. New York 1989. p177.

[3] L.C. Grove, Algebra, Academic Press. New York 1983. p95.

[4] O. Peron, Algebra II de Grayter Leipzig 1933 p35.

[5] W. Rudin, Principles of Mathematical Analysis. 3rd edition. Mc Graw –Hill 1974. p184.

[6] K. B. Yuguda (2004).Geometric Considerations of an Analogue of Rolle's Theorem for a Class of Polynomials in the Complex Plane. Bagale; Journal of Pure and Applied Science.