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On the coefficients of holomorphic functions and the identity

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Abstract

We give, in this note, a simple proof of the identity.

$$S_n - S_{n-1} = \binom{2n}{n},\ n \geq 1 \text{ where } S_n = \sum_{i+j+k=n} \binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k}$$

by relating it to the coefficients of the series expansion of a holomorphic function of several complex variables where the summation is taken over all non-negative integers i, j, k such that i + j + k = n.

Keywords: Holomorphic functions, series expansion of holomorphic functions, coefficients of series expansion, absolute terms of an identity.

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1.0 Introduction

Let $S_n = \sum_{i+j+k=n} \binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k}$, where the summation is taken over all non-negative integers *i, j, k* such that *i* + *j* + *k* = *n*. It is required to establish the identity.

$$S_n - S_{n-1} = \binom{2n}{n},\ n \geq 1 \tag{1.1}$$

Direct combinatorial arguments have often been the method of proof [cf 2]. This note employs the coefficients of the series expansion of a holomorphic function of several complex variables. The precise result to be employed is the following:

1.1 Theorem (Osgood)

If a complex-valued function f is continuous in an open domain $D \subseteq \not\subset^k$ and is holomorphic in each of the variables $z_1,\ z_2, \ldots z_k$ separately, then it is holomorphic in $\underline{z} = (z_1,\ z_2, \ldots z_k)$ and

$$f(\underline{z}) = \sum_{n_1, n_2, \ldots n_k = 0} a_{n_1, n_2, \ldots n_k} (z_1 - w_1)^{n_1} (z_2 - w_2)^{n_2} \ldots (z_k - w_k)^{n_k}$$

where

$$a_{n_1, n_2, \ldots n_k} = \left(\frac{1}{2\pi i}\right)^k \iint \ldots \int \frac{f(\varsigma_1, \varsigma_2, \ldots \varsigma_k) d\varsigma_1 d\varsigma_2 \ldots d\varsigma_k}{(\varsigma_1 - w_1)^{n_1+1} (\varsigma_2 - w_2)^{n_2+1} \ldots (\varsigma_k - w_k)^{n_k+1}}, \tag{1.2}$$

$(w_1,\ w_2, \ldots w_k) \in D;\ (\varsigma_1, \varsigma_2, \ldots \varsigma_k) \in |\varsigma_j - w_j| = r_j$ and *f* is a holomorphic function of ς_j in $|\varsigma_j - w_j| < r_j$.

2.0 We employ formula (1.2) to establish the identity (1.1) as follows

Write $\sigma_{n,i} = \sum_{i=0}^{n} \binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i}$, then

$$S_n = \sum_{i=0}^{n} \sigma_{n,i} \tag{2.1}$$

For a fixed i, $j + k = n - i$, hence $k = n - i - j$ and j varies between 0 and n – i with $j + k = n - i$, $k + i = n -$ j. Indeed, putting $i + j + k = n$, then $j + k = n - i$, $k = n - i - j$, $k + i = n - j$ and $\binom{k+i}{k} = \binom{k+i}{i} = \binom{n-j}{i}$.

For a fixed i, we have

$$\binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k} = \binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i};\ j = 0,1,\ldots n-1$$

and thus

$$S_n = \sum_{i=0}^{n}\ \sum_{i=0}^{n-i} \binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i} = \sum_{i=0}^{n} \sigma_{n,i}\ .$$

Consider the function $f(z, w)$ given by

$$f(z,w) = (1+z)^i (1+w)^i (2+z+w)^{n-i} = (1+z)^i (1+w)^i\ (1+z+1+w)^{n-1}$$

$$= (1+z)^n (1+w)^i \sum_{j=0}^{n-i}\binom{n-j}{j}\left(\frac{1+w}{1+z}\right)^j$$

$$= (1+w)^i \sum_{j=0}^{n-i}\binom{n-i}{j}\left(\frac{}{1+w}\right)^j \left(\frac{}{1+z}\right)^{-j}$$

$$= \sum_{j=0}^{n-i}\binom{n-i}{j}\left(\frac{}{1+w}\right)^{i+j} \sum_{m=0}^{n-j}\binom{n-j}{m} z^m$$

The coefficient of w^i in $f(z, w)$ is

$$\sum_{j=0}^{n-i}\binom{n-i}{j}\binom{i+j}{i}\sum_{m=0}^{n-j}\binom{n-j}{m} z^m$$

Hence, the coefficient of $w^i z^i$ in $f(z,w)$ is

$$\sum_{j=0}^{n-i}\binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i} = \sigma_{n,\ i}$$

Hence according to formula (1.2) we have

$$\sigma_{n,i} = \left(\frac{1}{2\pi i}\right)^2 \int_{|z|=r}\int_{|w|=r} \frac{f(z,w)\,dz\,dw}{z^{i+1}\,w^{i+1}}$$

Let $0 < r < 1$ and $\Gamma = C_r \times C_r$ where C_r: $|z| = r$, then

$$\sigma_{n,i} = \frac{-1}{4\pi^2}\int_\Gamma \frac{(1+z)^i (1+w)^i (2+z+w)^{n-i}}{(zw)^{i+1}}\ dzdw \tag{2.2}$$

Hence by (2.1) and (2.2) we have

$$S_n = \frac{-1}{4\pi^2}\int_\Gamma \left\{\sum_{i=0}^{n} \frac{(1+z)^i (1+w)^i (2+w+z)^{n-i}}{(zw)^{i+1}}\right\} dzdw$$

$$= \frac{-1}{4\pi^2}\int_\Gamma \frac{(2+z+w)^n}{zw} \sum_{i=0}^{n}\left\{\frac{(1+z)(1+w)}{(zw)(2+z+w)}\right\}^i dzdw$$

$$= \frac{-1}{4\pi^2} \int_\Gamma \frac{(2+z+w)^n}{zw} \left(\frac{1-\dfrac{(1+z)^{n+1}(1+w)^{n+1}}{(zw)^{n+1}(2+z+w)^{n+1}}}{1-\dfrac{(1+z)(1+w)}{(zw)(2+z+w)}} \right) dzdw$$

$$= \frac{-1}{4\pi^2} \int_\Gamma \frac{1}{(1+z+w)} \left\{ (2+z+w)^{n+1} - \left(\frac{(1+z)(1+w)}{(zw)} \right)^{n+1} dzdw \right\}$$

Hence, we have,

$$S_n - S_{n-1} = \frac{-1}{4\pi^2} \int_\Gamma \frac{1}{(1+z+w)(zw-1)} \left[\begin{array}{l} (2+z+w)^{n+1} - (2+z+w)^n - \left\{ \dfrac{(1+z)(1+w)}{zw} \right\}^{n+1} \\ + \left\{ \dfrac{(1+z)(1+w)}{zw} \right\}^n \end{array} \right] dzdw$$

$$= \frac{-1}{4\pi^2} \int_\Gamma \frac{1}{1-zw} \left\{ \frac{(1+z)^n (1+w)^n}{(zw)^{n+1}} - (2+z+w)^n \right\} dzdw$$

Since $|z| \le r < 1$, $|w| \le r < 1$, then $|zw| < 1$

Hence,

$$\frac{1}{1+zw} \text{ and } \frac{(2+z+w)^n}{1-zw}$$

are holomorphic in Γ and

$$\int_\Gamma \frac{(2+z+w)^n}{1-zw} dzdw = 0$$

Thus,

$$S_n - S_{n-1} = \frac{-1}{4\pi^2} \int_\Gamma \frac{(1+z)^n (1+w)^n (1-zw)^{-1}}{(zw)^{n+1}} dzdw$$

is the coefficient of $(zw)^n$ in the expansion of the function $(1 + z)^n (1 + w)^n (1 - zw)^{-1}$ in powers of z, w and is equal to $\sum_{k=0}^{n} \binom{n}{k}^2$.

Finally, consideration of absolute terms in the identity

$$(1+x)^n \left(1+\tfrac{1}{x}\right)^n = \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^{2n}$$

gives

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$

Hence, $$S_n - S_{n-1} = \binom{2n}{n}$$

3.0 Conclusion

The coefficients of the series expansion of a holomorphic function of several complex variables are employed to establish a combinatorial identity.

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