Journal of the Nigerian Association of Mathematical Physics
Volume **12 (May, 2008), 479 - 484**

A constrained optimal stochastic production planning model

Alfred C. Okoroafor and Godswill U. Achi
Department of Mathematics, Abia State University, Uturu.

Abstract

This paper considers the stochastic production planning problem with the constraint that the inventory level exceeds the demand over a planning period. Using the existence of a homogeneous Markov non-randomized optimal policy, the existence of a unique solution to the associated Hamilton-Jacobi-Bellman equation is established and the optimal policy is characterized. Also advocated is a stochastic iterative procedure for locating the optimal inventory level.

***Keywords*:** Stochastic differential equation, Hamilton-Jacobi-Bellman equation, inventory level, production control and Markvo policy.

1.0 Introduction

In production planning, one of the most unstable variables is the inventory level. This is influenced by certain unavoidable environmental uncertainties: like sudden random demand fluctuations, inventory spoilage, sales return etc. They make ideal production policy for a wide class of cost functional impossible (Besoussan et al, 1984 [3]).

To take care of these various sources of environmental randomness, we represent uncertainty by a filtered probability by n-dimensional Brownian motion *W*, defined on $(\Omega, \Im, P)$ and satisfying the usual condition, see for example (Barles, 1997 [2]). Thus, we move from deterministic problem to a stochastic one by considering the "noisy" environment in order to model their behavior fairly accurately by adding an additive noise term in the state dynamics. This takes care of the various sources of environmental randomness, see for example (Fleming, et al, 1993 [4])

The general form of production planning is then formulated by representing the inventory level by a stochastic process $\{X_t, t \geq 0\}$, defined on the probability space and generated by $\Im_t$ with an overall noise rate that is distributed like white noise, δdW_t, and whose dynamics is governed by the Ito stochastic differential equation

$$dX_t = (U_t - Z_t)dt + \delta dW_t \tag{1.1}$$

where δ is the intensity of the noise

For basic stochastic concepts see (Karatzas et al 1991 [6]) Z_t denotes the constant demand rate and U_t, the production function, is a non stochastic parameter controlled by the investor. The objective is to find an optimal control policy which minimizes the associated expected cost functional

$$V(t,x,u) = E\left[\int_0^\infty e^{-\rho t} c(t,x,u)dt / X_0 = x\right], \tag{1.2}$$

where $C(\cdot)$ is a cost function, see (Ghosh, et al, 1992 [5]) for background and some other references ρ > 0 is a discount factor and X_t is the solution of the stochastic differential equation (1.1). For some earlier work in this area see for example (Besoussan et al, 1984 [3], Ghosh et al 1992 [5])

In these models, the objective is to schedule production over the planning period so that demand is satisfied at minimum cost where demand is known in advance and no back ordering is allowed. Instead, in this paper in order to make an optimal production decision today, we assume that future cost of production may exceed the cost of current production plus inventory carrying cost so that it would be more profitable to produce more than the current periods demand and carry inventory forward to satisfy future demand.

2.0 Problem formulation

We first consider a general n-dimensional model and latter specialize to a dimensional case for which explicit solution is obtained. The general form of the stochastic production planning model we would to consider takes the form see(Bessoussan et all, 1984 [3])

$$dX_t = (U_t - Z_t)dt + \delta dW_t \tag{2.1}$$

where (2.1) is the dynamics of the X_t and subject to

$$V(t,x,u) = E\left[\int_0^{\infty} e^{-\rho t} c(t,u_t,x_t)dt / X_0 = x\right] \tag{2.2}$$

and $C(t,u_t,x_t)$ is a quadratic cost function, ρ is the diffusion coefficient which influences the average size of the fluctuation of X_t,

$$u_t = [u_1(t), u_2(t), \ldots u_n(t)]^T, \quad \sigma = diag(\sigma_1, \ldots \sigma)$$

and

$$w_t = [w_r(t), \ldots w_n(t)^T$$

is an *n*-dimensional standard Brownian motion and $[\cdot]^T$ denotes the transpose of $[\cdot]$. (2.1) has the following differential generator

$$L^u = f \;\; D_x + \tfrac{1}{2} T_r(\sigma\sigma^T) D_{xx}^2 \tag{2.3}$$

where T_r is the trace, $D_x = \dfrac{\delta}{\delta_x}$ and the $f = (U_t - Z_t)$ is the drift function, characterizing the local trend see for example (Kushner, 1967 [7]).

Without loss of generality, we assume that the initial inventory level is zero and the unit cost of production and the cost of holding inventory is one. We then wish to specify a production plan policy that minimizes the performance index.

$$V(t,x,u) = E\left[\int_0^{\infty} \left(X_t^T X_t + U_t^T U_t\right)dt\right] \tag{2.4}$$

The objective is to find an optimal control Markov policy $U(\cdot)$ which minimizes the expected quadratic objective functional (2.4) and takes a feed back form $U_t = U(t,x_t)$ for a suitably defined U. The optimal costs are then defined by

$$V^*(t,x) = \inf_u = \inf[V(t,x,u)] \tag{2.5}$$

2.1 Existence of optimal Policy

It is well known (Fleming, et al, 1993 [4]) that the function V^* can be characterized by dynamic programming principle as solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial V^*}{\partial t} + \inf_{u \geq 0}[L^u V^* + C(t,x_t,u_t)] = 0 \tag{2.6}$$

where L^u is the second order differential operator to the diffusion and defined by (2.3).

The existence of a homogeneous Markov optimal policy has been proved by (Ghosh, et, al, 1992 [5]) by a convex analytical method. Thus, by the verification theorem (Barles, 1987 [2]), there exists a minimizer

$$u_t^* \ of \ \inf_u [L^u V(t,x) + X_t^T X_t + U_t^T U_t] \tag{2.7}$$

Such that

$$\frac{\partial V^*}{\partial t} + L^{u^*} V + X_t^T X_t + U_t^T U_t = 0 \tag{2.8}$$

where $dX_s(s, X_s)$ is a well defined control process called the optimal Markov control process.

We consider a homogeneous version by assuming that the performance index $V(t,x,u)$ is independent of *w*ithout any loss of generality, so that upon application of the above principle of optimality, the dynamical programming equation associated with our optimal expected cost V^* is then

$$\rho V(x) = \inf_u \left\langle L^u V(x) + x^T x + u^T u \right\} \tag{2.9}$$

and has a unique solution given by $V^*(x) \inf_u [V(x,u)]$, where $V(x,u) = E\left[\int_0^\infty e^{-pt}(X^T X + U^T U)dt\right]$.

From (2.9); elementary calculation shows that the admissible control policy u^* which minimizer (2.7) takes the feedback form

$$u^* = U(x) = \frac{V_x^x}{2} \tag{2.10}$$

where $V_x(x) = \frac{\partial V}{\partial x}$ and the solution $V(x,)$ to (2.9) which gives the general value $V^*(x) i\,\underset{u}{nf}\, V(x,u)$ which satisfies the non-linear partial differential equation

$$\rho V = \frac{V_x^T V_x}{4} - Z^T V_x + \frac{1}{2} Tr(\sigma\sigma^T) V_{xx} + x^T x \tag{2.11}$$

To investigate this model quantitatively, we often resort to numerical techniques. See for example (Kushner, 1990 [7]). Despite the complexity of this highly non-linear partial differential equation (2.11). closed-form solution have been found in many interesting settings, see for example (Akella, et al 1986 [1]) Instead in this paper, we specialize in a two dimensional case for which explicit solution can be obtained and specify the following condition.

$$x = Z + \frac{B}{2} \tag{2.12}$$

that is, the inventory level exceeds the demand vector Z by $\frac{B}{2}$, where B is a given vector. Furthermore, we assume that V is quadratic since the cost of functional is quadratic see for example, (Bensossan, et al, 1984 [3]). We now state our main result.

***Theorem* 2.1**

Let V(x) denote the value function given by

$$V(x) = i\,\underset{u}{nf}\, E\left[\int_0^\infty e^{-pt}(x^T x + u^T u)dt\right]$$

and $dx = (u - z)dt + \sigma dw,$ *then there is an optimal admissible control* $U(x) = -\frac{V_x(x)}{2}$ *and* $V(x) = x^T mx + B^T x + r$ *solves the HJB equation* $\rho V(x) = \inf_u \langle L^n v(x) = x^T x + u^u u \rangle$ *for m* $= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $a = -2(1+\rho) \pm 2\sqrt{1+(1+\rho)^2} = b$ *and* $r = \frac{1}{\rho}\left[\frac{1}{2}T,(\sigma\sigma^T)m + \frac{1}{4}BB^T + Z^T B - \rho B\right.$

$\left.\left(Z + \frac{B}{2}\right)\right].$

Proof

Applying the dynamic programming approach, a little calculation gives the *U* that minimizes (2.9) to be (2.10) substituting (2.10) into (2.9) yields

$$\rho V(x) = -\frac{V_x^T V_x}{2} - Z^T V_x + \frac{1}{2}T_r(\sigma\sigma^T)V_{xx} + x^T x + \frac{V_x^T V_x}{4}$$

$$= -\frac{1}{4}V_x^T V_x - Z^T V_x + \frac{1}{2}T_r \sigma\sigma^T V_{xx} + x^T Ix$$

Then assuming the quadratic solution of the associated HJB equation to be $V(x) = x^T mx + B^T x + r,$ we have

$$\rho(x^T mx + B^T x + r) = \frac{1}{4}[(2mx + B^T)^T(2mx + B^T)] - Z^T(2mx + B^T) + \frac{1}{2}T_r(\sigma\sigma^T)(2m) + x^T Ix$$

$$\rho = x^T mx + \rho B^T x + \rho r = -x^T mx - B^T mx - \frac{1}{4}B^T B - 2mxZ^T - Z^T B - T_r(\sigma\sigma^T)m + x^T Ix$$

$$\rho r - xT_r(\sigma\sigma^T)m + \frac{1}{4}B^T B = [I - m^T m]xx^T - [B^T + 2Z^T]mx$$

$$-\rho x^T mx + \rho B^T x$$

$$= [I - m^T m - \rho m]xx^T - x^T mx + \rho B^T[I - m^T m - \rho m - m]xx^T + \rho B^T\left(Z + \frac{B}{2}\right)$$

Hence if

$$r = \frac{1}{\rho}\left[\frac{1}{2}T_r(\sigma\sigma^T)m + \frac{1}{4}B^T B + Z^T B - \rho B^T\left(Z + \frac{B}{2}\right)\right]$$

Then

$$[I - m^T m - (1+\rho)m] = 0$$

Since $m = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ solving for a and b we have

$$a = b = -2(1+\rho) \pm 2\sqrt{1+(1+\rho)^2}$$

Thus for given vector B and fixed discount rate $0 < \rho < 1$, the value function $V(x)$ that solves the HJB can be obtained in a closed form.

3.0 Analysis of the value function *V*(*x*)

We can see that for a fixed $\rho \succ 0$, the value function $V(x)$ is strictly convex and therefore there exists a unique x^* such that

$$V_x(x^*) = 0$$

In this case, form (2.10) we have that the optimal production is

$$U^* = \begin{cases} -V_x(x) & if \quad x \leq x^* \\ 0 & if \quad x \geq x^* \end{cases}$$

Since the point x^* is the minimum point of $V(X)$, therefore

$$V_x(x)^* = \begin{cases} \leq 0 & if \quad x \leq x^* \\ \geq 0 & if \quad x \geq x^* \\ = 0 & if \quad x = x^* \end{cases}$$

We can deduce that, for large inventory level, it is more profitable to stop production which for large stock it is optimal to produce at the highest rate possible. Thus, if the optimal feed back policy U_c is given by

$$U_c \begin{cases} 0 & if \quad V_x(x_c) \leq -2 \\ -V_x(x_x) & if \quad -Z < V_x(x_c) < 0 \\ c & if \quad V_x(x_c) \geq 0 \end{cases}$$

we can choose any

$$U_c \in [0, c]$$

but it becomes unprofitable to produce at

$$x_c > x^*$$

A stochastic iterative method due to (Okoroafor, 2006 [8]) provides a useful technique for computing the vector x^* when it exists.

4.0 Conclusion

We have analyzed the optimal control of a 2-dimensional constrained stochastic production planning model with discounted criterion on the infinite horizon. Also the existence of a closed frform solution to the associated HJB equation with quadratic growth is established and the optimal policy is characterized. This model is well suited in manufacturing industries where production is made to control immediate and meet future demand.

Reference

[1] Akella A. and Kumar P. R. (1986) Optimal Control of production rate in a failure prone mamuifacturing system . IEEE. Trans. Automatic control, AC-31, 116-126.

[2] Barles G. (1997) “Solution de Viscosity des equations de Hamilton Jacobi. Springer Verlag

[3] Besoussan A, Sethi S. P. Vickson R. and Derzko (1984) “Stochastic production planning with production constraints. SIAM J. Control and optimization, **No** 22 920-935.

[4] Fleming W. and M. Soner (1993) “Controlled Markov processes and viscosity solutions springer Verlag.

[5] Ghosh, M. K., Arapostathis and S> I Marcus (1992) Optimal Control of Switching diffusions with application to flexible manufacturing systems. SIAM J. conrol and optimization Vo. 30 30 **No** 6 1-23.

[6] Karatzas I. and S. E. Shreve (1991): Brownian motion and stochastic calculus Springer, New York 315

[7] Kushner H. J. (1967) “Optimal discounted stochastic control for diffusion Processes. SIAM J. control Vol 3, No. 4 520 – 531.

[8] Okoroafor A. C. (2006) “A strong convergent stochastic algorithm for the global solution of Autonomous linear equation in n-spaces. Journal of Applied sciences 6(11) 2506-2509 ©2006 Asian Network for Scientific information.