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Mark-resighting survey method for the estimation of population size in a closed animal population

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Abstract

The problem of estimating the population size of a closed population by method of mark –resighting sampling design was examined. We adopt the classical and Bayesian inference procedures related to the exact sampling distribution for our approximation. The maximum likelihood estimation (M.L.E) is derived for a case when N is large and it is observed that this estimator coincides with the M.L.E derived using the Binomial approximation.

***Keywords*:** Binomial Approximation, Capture-recapture models, Mark-Resighting Survey Maximum Likelihood Estimation.

1.0 Introduction

The problem of estimating the size of a closed population is of high interest in several wildlife population monitoring since the need to have a true knowledge of the total population estimate under care is important. However, the method of estimating the size of a closed population based on the results of a certain type of mark-resighting sampling design has been in use .This method is similar to the commonly used multiple capture-recapture design. The procedure involved in first tagging a number of randomly selected animals with an identifiable mark and latter randomly sighting them on several occasions and noting the number of marked animals. This type of sampling procedure is being used in several wildlife population monitoring studies with some animal population, this procedure is more economical and can be easily adopted. For the adoption of the basic capture-recapture model of estimating the size of a closed population, we define n_0 as the number of the randomly captured animals from the population which shall be returned back into the population, n_1 represent the number of animal captured at the second time, m_1 is the number of marked animals from the second recapture. The Lincoln-Peter estimate of the population size is given by $\hat{N} = n_0 n_1 / m_1$.

Regardless of whether the second sample is taken with or without replacement, this Lincoln-Peterson estimate is a biased estimate and nearly unbiased estimator are given in Ananda (1997 [5]) and Efron (1981 [9]). Approximate confidence intervals related to these nearly unbiased estimators are given in Jensen (1989 [7]) .There are lot of extension for capture –recapture method .A recent review of this are in Pollock (1991). One commonly used extension is the multiple captures recapture surveys. With multiple captures –recapture surveys the sampling scheme involves taking samples from the population, counting the number of tagged animals in the sample tagging previously untagged animals and returning the sample to the population. In some cases, sampling is done over a period of time and tagging animals at each stage could be very costly and time consuming .In order to avoid this problem, Wehausen (1992 [2]) used the following capture –recapture design. First tag n_0 randomly selected animals with visible *all correspondence to the author.

and identifiable marks and then take several second stage samples (say S samples), where the i_{th} sample $(i = 1\Lambda\ )$ has n_i animals and m_i marked animals. With large animals such as mountain sheep, these second stage samples may be taken by visually sampling from a helicopter. Furthermore each sample can be collected by searching the entire mountain range at once (without over looping the areas to avoid re-counting). Essentially this design is equivalent to taking s second stage independent samples, each sample being a random sample without replacement from the original population. Here, both n_i and m_i are both random variables and the distribution of m_i given n_i follows a hypergoemetric distribution. Jeager et.al (1991 [3]), (1994 [4]) has also used similar sampling design in their mountain sheep monitoring.

Ananda (1997 [5]), used the binomial approximation to solve this problem and he gave a point and interval estimators of N by putting a prior on $p=n_0/N$, he gave a Bayesian estimators of N and credible regions of N as well. In general, it is known that when $n_i < 0.1N$ the hyper geometric distribution can be approximated by the binomial distribution. However, in many cases, in particular if the second stage sample are based on a entire search of the mountain range, these second stage sample sizes n_i could be relatively large and the condition $n_i<0.1N$ may not hold. Moreover, when one uses the binomial distribution for the distribution of m_i given n_i, the sample design is equivalent to taking s second stage independent samples, each sample being a sample with replacement from the original population.

For the purpose of this study we consider the case where the sample size n_i $(i=i....s)$ are larger in comparison with N we construct point and interval estimators for N . The maximum like hood estimation (MLE) procedure of N is described when N is large by approximating the likelihood function, we derived a closed form formula for the MLE and as expected, this approximate MLE coincide with the M.L.E derived using the binomial approximation, The outline of this work is as follows section 2 gives simple model for the mark resighting design, a simple comparison of the Bayesian inference related to the exact and approximate like hood function were established in section 3. Finally a simple conclusion was drawn to mark the end of this study.

2.0 Model for the Mark-resighting sample design.

By considering the notations described in section 1, let us denote the parameter of interest, the total number of animals in the closed population, and n_o denotes the total number of tagged animals in the population. Suppose that s independent random samples are available from this population each sample being a random sample without replacement, n_i , and m_i $(i=i....s)$ denotes the number of animals and the number of marked animals in each sample respectively. Then the probability distribution of m_i given n_i follows the hype geometric distribution.

$$f(m_i|n_i) = \frac{\binom{n_0}{m_i}\binom{N-n_0}{n_i-m_i}}{\binom{N}{n_i}}, m_i = 0,1,\Lambda\ n_i \tag{2.1}$$

Obviously, the like hood function would be

$$L(N) = \prod_{i=1}^{S} g(n_i)\binom{n_0}{m_i}\binom{N-n_0}{n_i-m_i}\Big/\binom{N}{n_i} \tag{2.2}$$

We define $g(n)$ as the probability density for the second stage sample size and assume that the $g(n)$ does

not depend on the parameter N. One can evaluate the maximum likelihood estimator (M.L.E) of N numerically by maximizing the maximum likelihood function. However, for large, N one might get into some numerical difficulties, in particular calculations involving confidence interval, let us denote this numerically evaluated M.L.E by $\hat{N}_n$

Analytically, there is no close form solution for the MLE of N. However, if N is large, the MLE of N can be expressed in a closed form solution and it is given by

$$\hat{N}_a = n_0\left(\sum n_i\right)/\left(\sum m_i\right) \tag{2.3}$$

When $n_i < 0.1\,N$, the hyper geometric distribution given in (2.1) can be approximated by the binomial distribution. Using this approximation in the place of equation (2.1), the M.L.E of N is exactly the same as the approximates M.L.E given in equation, (2.3) statistically, this binomial approximation assumes that the sample observations in each sample are taken one at a time with replacement. We show the proof of (2.3) as follows:

Recall, from sterling's formula for large m we have, $\Gamma(m) \cong \sqrt{2\pi e^{-m} m^{m+0.5}}$ where m is a positive integer. Using this formula in equation (2.3), we have

$$L(N) \cong \prod_{i=1}^{S} c_i \frac{(N-n_i)^{N-n_i+0.5}(N-n_0)^{N-n_0+0.5}}{N^{N+0.5}(N-n_0-n_i+m_i)^{N-n_0-n_i+m_i+0.5}},$$

where $C_i = \dfrac{g(n_i)n_0!n_i!e^{-m_i}}{m_i!(n_0-m_i)!(n_i-m_i)!}$. By differentiating, $\ln\left(L(N)\right)$ one can see that the maximum of $L(N)$ occurs when

$$\sum_{i=1}^{s}\ln\left(1-\frac{n_i}{N}\right)-\sum_{i=1}^{s}\ln\left(1-\frac{n_i-m_i}{N-n_0}\right)+\sum_{i=1}^{s}\frac{n_i}{2N(N-n_i)}-\sum_{i=1}^{s}\frac{n_i-m_i}{2(N-n_i)(N-n_i-n_0+m_i)}=0$$

For large N, $\sum_{i=1}^{s}\dfrac{n_i}{2N(N-n_i)}-\sum_{i=1}^{s}\dfrac{n_i-m_i}{2(N-n_i)(N-n_i-n_0-m_i)} \cong 0.$ Since $\ln(1-x) \cong x \; (if \;\; x \to 0)$

we get, $\sum_{i=1}^{s}\dfrac{n_i}{N}-\sum_{i=1}^{s}\dfrac{n_i-m_i}{N-n_i} \cong 0$, which yields the approximate estimate in (2.3). Due to the complexity of the sampling of distribution, finding the sampling distribution of these two estimators or finding estimates for the variance of this estimator are difficult. Therefore, we use the Jackknife procedures discussed by Efron (1990 [9]) and Miller (1974) to construct approximate confidence intervals for N. Suppose $\hat{N}_{(i)}(i=1\,2\,......\,s)$ be the estimate of N when the i_{th} sample (n_i, m_i) is omitted from the sample $\{(n_i, m_i), \quad i=1\,2....s\}$. This $\hat{N}_{(i)}$ stands for the numerical M.L.E $\hat{N}_n$ or the approximate MLE $\hat{N}_a$ (When the i_{th} sample is omitted). Then the i_{th} "pseudo-value is defined as; $J_i = s\hat{N}-(s-1)\hat{N}_{(i)}$, where $\hat{N}$ is the estimate obtained by using the complete sample. Then an approximate $100(1-\alpha)\%$ confidence interval for N (see Miller, (1974 [10]) for details) is given by:

$$\left(J\left(\hat{N}\right) - t_{\frac{\alpha}{2}, n-1}\, \delta_{j\left(\hat{N}\right)},\; J\left(\hat{N}\right) + t_{\frac{\alpha}{2}, n-1}\, \delta_{j\left(\hat{N}\right)} \right),$$

where $J\ (\hat{N}) = \sum_{i=1}^{s} J_i / s$, $\delta_{j(\hat{N})} = \sqrt{\dfrac{\sum_{i}^{s} (J_i - J(\hat{N})^2}{(S-1)}}$ and $t_{(\alpha/2),\, s-1}$ is the $(1-\alpha/2)\, t_h$ quintile of the t-distribution with the degrees of freedom $s-1$.Under certain conditions, the hyper geometric distribution can be approximated by the Poisson distribution Johnson et.al ,(1992 [11]), Smith, (1988 [12]) used the Poisson distribution to analyze certain types of capture - recapture data. With the mark-resighting model discussed in this paper; we could use the Poisson approximation to analyze the data as follows: As in Castledine (1981 [13]), when n_i values are large and n_o/N is small, by approximating the hyper geometric distribution given in (2.1) by the Poisson distribution with parameter $\lambda_I = n_0 n_i / N$ the likelihood function can be written as

$$L(N) = \left(\prod_{i=1}^{s} g(n_i)\right)\left(\prod_{i=1}^{s} m_i!\right)^{-1} n_i^{\,m_i} n_0^{\,m_i} N^{-1} \sum_{i=1}^{s} \exp\left(-\sum_{i=1}^{s} n_i n_0 \Big/ N\right) \qquad (2.4)$$

In this case, it is easy to show that the M.L.E of N is same as the approximate MLE given in (2.3). Therefore, for confidence intervals, the Jackknife procedure will produce the same confidence interval as in $\hat{N}_a$.

3.0 Bayesian inference related to the exact and approximate like-hood functions

In this section, we shall describe some Bayesian inferences related to the exact and approximate likelihood functions described in (2.2) and (2.5). In a similar capture-recapture survey design. Smith (1988 [12]), used a gamma prior density to model prior information regarding $w = 1/N$. He used the gamma density $g(w) = b^a w^{a-1} e^{-bw} / \Gamma_{(a)}$ on $w = 1/N$ where the constants $a > 0$ and $b > 0$ are chosen to reflect the strength of historical data. In another similar design, Castledine (1981), used a beta prior density to model prior information regarding $p = n_0/N$.Using the binomial approximation to the hyper-geometric distribution. Ananda (1997 [5]), looked at the Mark – re sighting survey described in this paper in a Bayesian frame work and used the beta prior density to model prior information regarding $p = n_0/N$.However, the binomial approximation requires that the second stage sample sizes to be very small in order to have independent samples, which may not be true with many applications.

As in Smith (1988 [12]), we use a discrete version of a gamma prior on N to reflect the prior information with our mark – resighting scheme, suppose that the prior density on N is proportional to

$$\pi(N) \sim (1/N)^{a-1} e^{-b/N} \text{ For } N = n_0, n_0 + 1, n_0 + 2$$

where a and b are two non-negative constants chosen to reflect the prior information. These constants must be evaluated using prior information when prior information is not available, one can use a non informative prior by choosing $a = 1, b = 0$ with $\pi(N) = c$ for $N = n_0, n_0 + 1, n_0 + 2.....$ where c is a constant. The posterior density of N is given by

$$\pi(N/data) = \frac{\left(\frac{1}{N}\right)^{a-1} e^{-(b/N)} \prod_{i=1}^{s} \left\{ \binom{n_0}{m_i} \binom{N-n_0}{n_i-m_i} \Big/ \binom{N}{n_i} \right\}}{\sum_{N=k_0}^{\infty} \left(\frac{1}{N}\right)^{a-1} e^{-(b/N)} \prod_{i=1}^{s} \left\{ \binom{n_0}{m_i} \binom{N-n_0}{n_i-m_i} \Big/ \binom{N}{n_i} \right\}} \tag{3.2}$$

where $m_i = 0\,1....,n_i \quad (i = 1.....k) \quad N \geq k_0$ and $k_0 = \max_{1\leq i\leq k}(n_i, n_0 + n_i - m_i)$

Assuming the quadratic loss function, the Bayesian estimator of N is given by (for details on Bayesian calculation, see Berger (1988 [14]),

$$\hat{N}_{B1} = \frac{\sum_{N=n_0}^{\infty} \left(\frac{1}{N}\right)^{a-2} e^{-(b/\ N)} \prod_{i=1}^{s} \left\{ \binom{n_0}{m_i} \binom{N-n_0}{n_i-m_i} \Big/ \binom{N}{n_i} \right\}}{\sum_{N=k_0}^{\infty} \left(\frac{1}{N}\right)^{a-1} e^{-(b/\ N)} \prod_{i=1}^{s} \left\{ \binom{n_0}{m_i} \binom{N-n_0}{n_i-m_i} \Big/ \binom{N}{n_i} \right\}} \tag{3.3}$$

Numerically, this estimator can be evaluated by setting an initial larger upper limit for the sum and then gradually increasing it until the estimate is relatively stable. A programming code written in C++ to calculate this estimate is given in Smith (1988). A $100(1-\alpha)\%$ credible interval for N is given by $(\pi_{\alpha/2}, \pi_{1-\alpha/\ 2})$ where π_a is the a_{th} quantile of the posterior distribution given in eqn. (3.2). However, numerically getting an accurate numerical answer involves lots of calculations. Therefore, we propose the following approximation procedure which is based on the Poisson approximation to the
Hyper geometric distribution. When the condition for the Poisson approximation are correct, one can get the Bayesian inference related to the likelihood (2.5) as follows: When

$$\pi(p) = \frac{p^{a-1} e^{-bp}}{\int_0^{1/\ n_0} p^{a-1} e^{-bp}\, dp}, \quad if \quad 0 < p < 1/n_0 \tag{3.4}$$

Here a and b are two constants which depends on prior data. Again, if prior data is not available, one can use $a = 1 \quad and\ b = 0$ which reflect the non informative prior. Then the posterior density of p is

$$\pi(p|data) = \frac{e^{-p\left(b+n_0\sum_{i=1}^{s} n_i\right)} p^{\sum_{i=1}^{s} m_i + a - 1}}{\int_0^{\frac{1}{n_0}} e^{-p\left(b+n_0\sum_{i=1}^{s} n_i\right)} p^{\sum_{i=1}^{s} m_i + a - 1}\, dp} \tag{3.5}$$

If $0 < p < \dfrac{1}{n_0}$. Assuming a quadratic loss, one can show that the Bayesian estimate of N is given by

$$\hat{N}_{B2} = \frac{\left(b + n_0 \sum_{i=1}^{s} n_i\right) F\left(\frac{b}{n_0} + \sum_{i=1}^{s} n_i ; a + \sum_{i=1}^{s} m_i\right)}{\left(a - 1 + \sum_{k=1}^{s} m_i\right) F\left(\frac{b}{n_0} + \sum_{i=1}^{s} n_i ; a - 1 + \sum_{i=1}^{s} m_i\right)} \tag{3.6}$$

Where $F(t;a)$ is the cumulative distribution function of the gamma distribution with parameter a , *i.e.*

$$F(t;a)=\int_0^t \frac{e^{-x}x^{a-1}}{\Gamma(a)}dx \tag{3.7}$$

Since the gamma distribution is readily available in any statistical software package forward and easy. Let us define the α_{th} quantile of the gamma distribution with parameter a by $I[\alpha;a]$, i.e. $F(FI[\alpha;a];a)=\alpha$ it can be shown that a $100(1-\alpha)\%$ confidence interval for N is given by:

$$\left(\frac{\left(b+n_0\sum_{i=1}^{s}n_i\right)}{I\left[\left(1-\frac{\alpha}{2}\right)F\left(\left(\frac{b}{n_0}+\sum_{i=1}^{s}n_i\right)a+\sum_{i=1}^{s}m_i\right);a+\sum_{i=1}^{s}m_i\right]I\left[\frac{\alpha}{2}F\left(\left(\frac{b}{n_0}+\sum_{i=1}^{s}n_i\right)a+\sum_{i=1}^{s}m_i\right)a+\sum_{i=1}^{s}m_i\right]}\right),$$

4.0 Conclusion

In the light of the study so far we observed that if correct prior is available the exact Bayesian method would be very useful and applicable in estimating our parameters, however if prior information is not available the use of N_{B1} given in equation (3.3) with the non informative choice $a=1$ and $b=0$ would give a better approximation.

References

[1] N.T.J. Bailey, improvements in the interpretation of recapture data, Journal of Animal Ecology 21 (1951) 120-127.

[2] J.D Wehausen, Demographic studies of mountain sheep in the Mojave Desert. Report IN, California Department of Fish and Game, Bishop 1992.

[3] J.R. Jaeger, J.D. Wehausen, V.C Bleich, Evaluation of the time lapse photography to estimate population parameters, Desert Bighorn Council Transaction 35 (1991) 5-8.

[4] J.R Jaeger, Demography and Movement of desert-dwelling mountain sheep in the Kingston and Clark Mountain Ranges, California, M. S Thesis, Department of Biological Sciences University of Nevada, Las Vegas, 1994.

[5] M.M.A Ananda, Bayesian methods for mark-resighting surveys, communications in statistics- Theory and Methods 26 (3) (1997). 685-697

[6] D.G Chapman, some properties of the Hyper geometric Distribution with Application to Zoological Censuses, Vol. 1, University of California publications in statistics, 1951, pp 131-160.

[7] A. L. Jensen, Confidence Intervals for nearly unbiased estimators in single- Mark and single recapture experiments, Biometrics 45 (1989) 1233-1237.

[8] G.A.F. Seber, The estimation of Animal Abundance, second ed., Macmillan, New York (1982)

[9] B. Efron, Nonparametric estimates of standard error: the Jackknife, the bootstrap, and other resampling plans, Biometrical (1981).

[10] Miller, The Jackknife – a review, Biometrical 61 (1974) 1-15.

[11] N.L. Johnson, S Kotz, A.W Kemp, Univariate Discrete Distribution, second ed., Wiley, New York 1992.

[12] P. J. Smith, Bayesian methods for multiple capture-recapture surveys, Biometrics 44 (1988). 1177-1189.

[13] B. J. castle dine, A Bayesian analysis of multiple-recapture sampling for a closed population, Biometrical 67 (1981) 197-210

[14] J.O Berger, statistical Decision Theory and Bayesian Analysis, second ed., Springer, New York, 1988.

[15] H. He, population size estimation from Mark- resighting surveys, M. S Thesis, Department of Mathematical Sciences, University of Nerada, Las Vegas, 1998.